MODIFIED BESSEL FUNCTIONS IN ANALYTIC NUMBER THEORY

BRUCE C. BERNDT, ATUL DIXIT, RAJAT GUPTA, ALEXANDRU ZAHARESCU

In Memory of Richard A. Askey

ABSTRACT. The modified Bessel functions $K_\nu(z)$, or, for brevity, K-Bessel functions, arise at key places in analytic number theory. In particular, they appear in beautiful arithmetic identities. A survey of these arithmetical identities and their appearances in number theory is provided.

1. REFLECTIONS OF THE FIRST AUTHOR

I became acquainted with Richard Askey at the University of Wisconsin while writing my Ph.D. thesis [6], wherein Bessel functions play a prominent role. Our friendship began either when I sought help from him about Bessel functions, or when (more probably) he learned that I was heavily involved with Bessel functions and wanted to see what I was doing with them. Although I cannot pinpoint any particular facts about Bessel functions that I learned from Dick, what I learned was substantial, and he certainly pointed me to Watson's treatise [81] to learn more. In summary, our friendship began in 1965 during the writing of my doctoral dissertation.

In five of the six chapters of my thesis, the ordinary Bessel functions $J_\nu(x)$ are featured, while in Chapter 6, the modified Bessel functions $K_\nu(x)$ are the focus. The first paper that I published appeared in the *Proceedings of the Edinburgh Mathematical Society*, and it was based on Theorem 6.1 in Chapter 6 in my thesis. In the following survey describing those instances in analytic number theory or related analysis wherein the Bessel functions $K_\nu(x)$ appear, Theorem 7.1 is a generalization of Theorem 6.1 in my thesis [6].

2. INTRODUCTION

The ordinary Bessel function $J_\nu(z)$, the Bessel function $Y_\nu(z)$ of the second kind, the Bessel function of imaginary argument $I_\nu(z)$, and the modified Bessel function $K_\nu(z)$ are defined, respectively, by [81, pp. 40, 64, 77, 78]

$$J_\nu(z) := \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}z\right)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \quad z \in \mathbb{C}, \tag{2.1}$$

$$Y_\nu(z) := \frac{J_\nu(z)(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad z \in \mathbb{C}, \nu \notin \mathbb{Z}, \tag{2.2}$$

$$Y_n(z) := \lim_{\nu \to n} Y_\nu(z), \quad n \in \mathbb{Z}, \tag{2.3}$$

$$I_\nu(z) := \sum_{n=0}^\infty \frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \quad z \in \mathbb{C}, \tag{2.4}$$

$$K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \quad z \in \mathbb{C}, \nu \notin \mathbb{Z}, \tag{2.5}$$

$$K_n(z) := \lim_{\nu \to n} K_\nu(z), \quad n \in \mathbb{Z}. \tag{2.6}$$

As $z \to \infty$, $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$, and $K_\nu(z)$ possess the asymptotic formulas [81, pp. 199, 202],

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O\left(\frac{1}{z}\right) \right), \tag{2.7}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O\left(\frac{1}{z}\right) \right), \tag{2.8}$$

$$I_\nu(z) = \sqrt{\frac{1}{2\pi z}} e^z \left(1 + O\left(\frac{1}{z}\right) \right), \tag{2.9}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + O\left(\frac{1}{z}\right) \right). \tag{2.10}$$

Both $J_\nu(z)$ and $K_\nu(z)$ arise in several problems and identities in analytic number theory. In particular, as evinced by the cosine-function appearing in (2.7), ordinary Bessel functions appear in certain problems involving circular, spherical, or cylindrical symmetry. The rapid decay of modified Bessel functions, as indicated in (2.10), point to possible appearances in summation formulas and contexts where theta functions might otherwise appear. As our title indicates, in this paper we devote attention to problems and identities in number theory wherein modified Bessel functions appear.

Throughout this paper, $\sigma = \text{Re}(s)$.

3. THE DIRICHLET DIVISOR PROBLEM

Let $d(n)$ denote the number of positive divisors of the positive integer n. Set

$$D(x) := {\sum_{n \leq x}}' d(n),$$

where the prime ′ on the summation sign indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted. Historically, perhaps the first instance in number theory in which modified Bessel functions appear is in the "Voronoï Identity," first proved by G. F. Voronoï [78] in 1904. For $x > 0$, it is given by

$$D(x) = x \left(\log x + 2\gamma - 1\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{x}{n}\right)^{1/2} M_1(4\pi\sqrt{nx}), \tag{3.1}$$

where γ denotes Euler's constant and $M_\nu(z)$ is defined by

$$M_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z). \tag{3.2}$$

Define the 'error term' $\Delta(x)$ by

$$D(x) = x \left(\log x + 2\gamma - 1\right) + \frac{1}{4} + \Delta(x). \tag{3.3}$$

Observe that $\Delta(x)$ is represented by the series of Bessel functions on the right-hand side of (3.1). Finding the order of magnitude of $\Delta(x)$ as $x \to \infty$ is the famous, unsolved, notoriously difficult *Dirichlet Divisor Problem*. It is conjectured that for every $\epsilon > 0$,

$$\Delta(x) = O(x^{\frac{1}{4}+\epsilon}), \qquad x \to \infty.$$

Most attempts at finding upper bounds for $\Delta(x)$ as $x \to \infty$, in essence, depend on bounding partial sums of the infinite series in (3.1). From (2.8) and (2.10), we see that the asymptotic formula (2.8) is a key ingredient in bounding $\Delta(x)$. Currently, the best result is due to M. N. Huxley [44], who proved that

$$\Delta(x) = O(x^{131/416+\epsilon}),$$

for each $\epsilon > 0$. Observe that $\frac{131}{416} = .3149\ldots$. A survey of much of the work that has been devoted to the *Dirichlet Divisor Problem* can be found in [18].

Other proofs of (3.1) have been given by A. Ivić [48] and S. Egger né Endres and F. Steiner [36], A survey article on the life and work of Voronoï has been written by J. Steuding [75].

4. AN IDENTITY OF RAMANUJAN FROM HIS LOST NOTEBOOK

On a page published with his lost notebook [71, p. 335], Ramanujan recorded two identities involving doubly infinite series of Bessel functions. The first is associated with the classical *Circle Problem* of Gauss, while the second is connected with the *Dirichlet Divisor Problem*. To state the latter formula as Ramanujan recorded it, first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer}, \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer}, \end{cases} \tag{4.1}$$

where $[x]$ is the integral part of x.

Theorem 4.1. *If $F(x)$ is defined by* (4.1)*, then, for $x > 0$ and $0 < \theta < 1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta))$$

$$+ \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left(\frac{M_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{M_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right), \tag{4.2}$$

where $M_1(z)$ is defined by (3.2).

For simplicity, assuming that x is not an integer, by an elementary argument, we observe that

$$D(x) = \sum_{n \leq x} \sum_{d|n} 1 = \sum_{dj \leq x} 1 = \sum_{d \leq x} \sum_{1 \leq j \leq x/d} 1 = \sum_{d \leq x} \left[\frac{x}{d}\right]. \tag{4.3}$$

Setting $\theta = 0$ in (4.2), we see that the left side of (4.2) is a generalization of (4.3), while the right-hand side contains a two-variable analogue of the series on the right side of (3.1). We conjecture that Ramanujan established (4.2) in order to attack the *Dirichlet Divisor Problem*, but he did not leave a record of any motivation or attempted proof. The only proof of (4.2) has been given by Junxian Li, and the first and fourth authors, and is difficult [20].

Extending the ideas of [20], the first author, Martino Fassina, Sun Kim, and the fourth author [16] established identities for certain finite sums of trigonometric functions in terms of Bessel functions. We provide one example [16, p. 6, Theorem 2.4].

Theorem 4.2. *Let $M_1(z)$ be defined by* (3.2). *If $0 < \theta, \sigma < 1$ and $x > 0$, then*

$$\sideset{}{'}\sum_{nm \leq x} \cos(2\pi n\theta)\cos(2\pi m\sigma)$$

$$= \frac{1}{4} + \frac{\sqrt{x}}{4} \sum_{n,m \geq 0} \left\{ \frac{M_1(4\pi\sqrt{(n+\theta)(m+\sigma)x})}{\sqrt{(n+\theta)(m+\sigma)}} + \frac{M_1(4\pi\sqrt{(n+1-\theta)(m+\sigma)x})}{\sqrt{(n+1-\theta)(m+\sigma)}} \right.$$

$$\left. + \frac{M_1(4\pi\sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{M_1(4\pi\sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}} \right\}.$$

5. The Voronoï Summation Formula

In his two long papers [78], Voronoï established a more general result than (3.1), which is called the *Voronoï Summation Formula*. If $f(x)$ is a suitable function, then Voronoï proved that

$$\sum_{j=0}^{\infty} d(j)f(j) = \int_0^{\infty} (2\gamma + \log x)f(x)dx$$

$$+ \sum_{n=1}^{\infty} d(n) \int_0^{\infty} f(x) \left\{ K_0(4\pi\sqrt{nx}) - \frac{1}{2}\pi Y_0(4\pi\sqrt{nx}) \right\} dx, \qquad (5.1)$$

where $Y_0(z)$ and $K_0(z)$ are defined by (2.3) and (2.6), respectively, and γ denotes Euler's constant. Alternative versions of (5.1) can be established in which the sums and integrals over $[0, \infty)$ are replaced by finite intervals. Voronoï's summation formula is reminiscent of the Poisson summation formula, and, indeed, proofs of (5.1) using the Poisson summation formula have been given; e.g., see [55] and [43]. An especially short proof of (5.1) was given by N. S. Koshliakov [51] in 1929. Several applications of (5.1) were made in a series of papers by him.

We offer a special case of (5.1) in the spirit of (3.1).

K. Soni [74] and, subsequently, also S. Yakubovich [84, p. 379] showed that, for $y > 0$,

$$\sum_{n=1}^{\infty} \left\{ K_0(2\sqrt{yn}) - 2y \frac{\log(4\pi^2 n/y)}{(4\pi^2 n)^2 - y^2} \right\} = \frac{1}{8} \log\left(\frac{4\pi^2}{y}\right) - \frac{\log\sqrt{y}}{y} - \frac{y}{4}. \qquad (5.2)$$

However, this identity was first established by Voronoï [78, Equations (5), (6)]. It also appears in Ramanujan's Lost Notebook [71, p. 254].

Another special case is given by Koshliakov's formula, given later in Theorem 13.2.

Define the generalized divisor function $d_k(n)$ by

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \sigma > 1.$$

In particular, M. Jutila [49], A. Ivić [47], and X. Li [57] examined the ternary divisor problem, i.e., finding bounds for the 'error term' for

$$\sum_{n \le x} d_3(n).$$

To study this problem, Jutila [49] developed an analogue of (5.1).

There now exist in the literature many analogues and generalizatioins of the Voronoï summation formulas, established under a variety of conditions and in vastly different settings. For example, see papers by E. Assing [1]; E. Assing and A. Corbett [2]; D. Banerjee, E. M. Baruch, and D. Bump [3]; D. Banerjee, E. M. Baruch, and E. Tenetov [4]; B. C. Berndt [10]; S. Bettin and J. B. Conrey [21]; A. Corbett [26]; A. Dixit, B. Maji and A. Vatwani [33]; D. Goldfeld and X. Li [37], [38], [39]; A. Ichino and N. Templier [45]; E. M. Kıral and F. Zhou [50]; N. S. Koshliakov [52]; A. G. F. Laurinčikas [56]; T. Meurman [59]; S. D. Miller and W. Schmid [62], [60], [61], [62]; S. D. Miller and F. Zhou [63]; Z. Qi [69]; N. Templier [77]; C. Nasim [65]; T. Watanabe [79]; T. A. Wong [82]; S. Yakubovich [83], [85]; V. Zacharovas [86]; and F. Zhou [89].

6. AN ANALOGUE OF THE THETA-INVERSION FORMULA

Consider a class of arithmetical functions studied by K. Chandrasekharan and R. Narasimhan [23]. Let $a(n)$ and $b(n)$, $1 \le n < \infty$, be two sequences of complex numbers, not identically 0. Set

$$\varphi(s) := \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s}, \quad \sigma > \sigma_a; \qquad \psi(s) := \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s}, \quad \sigma > \sigma_a^*, \tag{6.1}$$

where $\sigma = \operatorname{Re}(s)$, $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences of positive numbers, each tending to ∞, and σ_a and σ_a^* are the (finite) abscissae of absolute convergence for $\varphi(s)$ and $\psi(s)$, respectively. Assume that $\varphi(s)$ and $\psi(s)$ have analytic continuations into the entire complex plane $\mathbb{C}$ and are analytic on $\mathbb{C}$ except for a finite set $\mathbf{S}$ of poles. Suppose that for some $\delta > 0$, $\varphi(s)$ and $\psi(s)$ satisfy a functional equation of the form

$$\chi(s) := (2\pi)^{-s} \Gamma(s) \varphi(s) = (2\pi)^{s-\delta} \Gamma(\delta - s) \psi(\delta - s). \tag{6.2}$$

Chandrasekharan and Narasimhan proved that the functional equation (6.2) is equivalent to the 'modular' relation

$$\sum_{n=1}^{\infty} a(n) e^{-\lambda_n x} = \left(\frac{2\pi}{x} \right)^{\delta} \sum_{n=1}^{\infty} b(n) e^{-4\pi^2 \mu_n / x} + P(x), \qquad \operatorname{Re}(x) > 0, \tag{6.3}$$

where

$$P(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} (2\pi)^z \chi(z) x^{-z} dz,$$

where $\mathcal{C}$ is a curve or curves encircling all of $\mathbf{S}$.

Chandrasekharan and Narasimhan also proved that (6.2) and (6.3) are equivalent to the following identity for Riesz sums appearing on the left-hand side below. Let $x > 0$ and $\rho > 2\sigma_a^* - \delta - \frac{1}{2}$. Then the functional equation (6.2) is equivalent to the *Riesz sum* identity

$$\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \le x}{}' a(n)(x - \lambda_n)^{\rho} = \left(\frac{1}{2\pi} \right)^{\rho} \sum_{n=1}^{\infty} b(n) \left(\frac{x}{\mu_n} \right)^{(\delta + \rho)/2} J_{\delta + \rho}(4\pi \sqrt{\mu_n x}) + Q_{\rho}(x), \tag{6.4}$$

where the Bessel function $J_{\nu}(z)$ is defined in (2.1) and the prime $'$ on the summation sign on the left side indicates that if $\rho = 0$ and $x \in \{\lambda_n\}$, then only $\frac{1}{2} a(x)$ is counted. Furthermore, $Q_{\rho}(x)$ is defined by

$$Q_{\rho}(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\chi(z)(2\pi)^z x^{z+\rho}}{\Gamma(\rho + 1 + z)} dz, \tag{6.5}$$

where $\mathcal{C}$ is a curve or curves encircling $\mathbf{S}$.

In other words, if $a(n)$ and $b(n)$ are two sequences of arithmetical functions generated by Dirichlet series satisfying one of (6.2), (6.3), or (6.4), then the remaining two relations hold.

The arithmetical functions $a(n)$ and $b(n)$ satisfying (6.2) and (6.3) also satisfy a 'modular' relation in which the exponential functions are, roughly, replaced by modified Bessel functions. More precisely, the present authors proved the following theorem [14, Theorem 4.1].

Theorem 6.1. *Let* $\operatorname{Re}(\nu) > -1$ *and* $\operatorname{Re}(c), \operatorname{Re}(r) > 0$. *Assume that the integral below converges absolutely. Then,*

$$\frac{1}{2\pi r} \sum_{n=1}^{\infty} \frac{a(n)}{(c^2 + \lambda_n)^{(\nu-1)/2}} K_{\nu-1}(4\pi r \sqrt{c^2 + \lambda_n})$$

$$= \frac{1}{2\pi r^{\nu} c^{\nu-\delta-1}} \sum_{n=1}^{\infty} \frac{b(n)}{(r^2 + \mu_n)^{(\delta-\nu+1)/2}} K_{\delta+1-\nu}(4\pi c \sqrt{r^2 + \mu_n})$$

$$+ \int_0^\infty Q_0(x)(c^2+x)^{-\nu/2} K_\nu(4\pi r \sqrt{c^2+x}) dx, \tag{6.6}$$

where $Q_0(x)$ is defined by (6.5).

The authors also established a slightly more general theorem, which we do not give here. The first author's paper contains the first statement and proof of Theorem 6.1 [9, pp. 343–344]. One can also prove Theorem 6.1 by using the Voronoï summation formula (5.1) [12], [10, p. 154].

If we let $\nu = 1/2$ and appeal to the special case [81, pp. 79, 80]

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \tag{6.7}$$

after some elementary manipulation, we obtain the identity

$$\sum_{n=1}^\infty a(n) e^{-4\pi r \sqrt{c^2+\lambda_n}} = 2^{3/2} c^{\delta+1/2} r \sum_{n=1}^\infty \frac{b(n)}{(r^2+\mu_n)^{(\delta+1/2)/2}} K_{\delta+1/2}(4\pi c \sqrt{r^2+\mu_n})$$
$$+ 2\pi r \int_0^\infty Q_0(x) \frac{e^{-4\pi r \sqrt{c^2+x}}}{\sqrt{c^2+x}} dx,$$

which might be compared with (6.3).

The beautiful asymmetry of the modular relation (6.3) also characterizes (6.6) and other K-Bessel function identities in our survey.

We offer two examples to illustrate (6.6). Details for each example can be found in [14]. Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares, where representations with different orders and different signs in the summands are regarded as distinct. For example, if $k = 2$,

$$5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2,$$

and so $r_2(5) = 8$. In the notation of (6.1) and (6.2), $\delta = k/2$ and $\lambda_n = \mu_n = n/2$. If we further define $r_k(0) = 1$ and replace ν by $\nu + 1$, then (6.6) yields the identity

$$\sum_{n=0}^\infty \frac{r_k(n)}{(c^2+n/2)^{\nu/2}} K_\nu(4\pi r \sqrt{c^2+n/2})$$
$$= \frac{1}{r^\nu c^{\nu-k/2}} \sum_{n=0}^\infty \frac{r_k(n)}{(r^2+n/2)^{(k/2-\nu)/2}} K_{k/2-\nu}(4\pi c \sqrt{r^2+n/2}). \tag{6.8}$$

The identity (6.8) was first proved by the first author, Y. Lee, and J. Sohn [19, p. 39, Equation (5.5)]. The special case $k = 2$ was first proved by A. L. Dixon and W. L. Ferrar [34, p. 53, Equation (4.13)] in 1934. Another proof for $k = 2$ was given by Oberhettinger and Soni [66, p. 24].

Second, recall that Ramanujan's arithmetical function $\tau(n)$ is defined by

$$\sum_{n=1}^\infty \tau(n) q^n := \Delta(z) = q \prod_{n=1}^\infty (1-q^n)^{24}, \tag{6.9}$$

where $q = e^{2\pi i z}$ with $z \in \mathbb{H}$, that is $|q| < 1$. In the notation of (6.1) and (6.2), $\lambda_n = \mu_n = n$ and $\delta = 12$. Applying Theorem 6.1 and replacing ν by $\nu + 1$, we deduce that, for $\mathrm{Re}(\nu), \mathrm{Re}(c), \mathrm{Re}(r) > 0$,

$$\sum_{n=1}^\infty \frac{\tau(n)}{(c^2+n)^{\nu/2}} K_\nu(4\pi r \sqrt{c^2+n}) = \frac{1}{r^\nu c^{\nu-12}} \sum_{n=1}^\infty \frac{\tau(n)}{(r^2+n)^{(12-\nu)/2}} K_{12-\nu}(4\pi c \sqrt{r^2+n}). \tag{6.10}$$

Details for the proof of (6.10) can be found in either [14] or [19], where the first proof of (6.10) was given.

Reference [14] also contains examples of (6.6) for $\sigma_k(n)$, i.e.,

$$\sigma_k(n) := \sum_{d|n} d^k; \tag{6.11}$$

both even and odd primitive characters $\chi(n)$; and $F(n)$, the number of integral ideals of norm n in an imaginary quadratic number field.

7. Another Identity that Is Reminiscent of the Modular Relation (6.3)

Theorem 7.1 below is an analogue of Theorem 6.1. Theorem 7.1 and a generalization of it were established in [13, Theorems 4.1, 3.1].

Theorem 7.1. *Assume that* $\mathrm{Re}(\nu) > 0$ *and* $\mathrm{Re}(s) > 0$. *Also assume that* $\delta + \nu + 1 > \sigma_a^* > 0$. *Suppose that the integral on the right side below converges absolutely. Then,*

$$\frac{2}{s} \sum_{n=1}^{\infty} a(n)\lambda_n^{(\nu+1)/2} K_{\nu+1}(s\sqrt{\lambda_n}) = 2^{3\delta+\nu+1} s^\nu \pi^\delta \Gamma(\delta+\nu+1) \sum_{n=1}^{\infty} \frac{b(n)}{(s^2+16\pi^2\mu_n)^{\delta+\nu+1}}$$
$$+ \int_0^{\infty} Q_0(x) x^{\nu/2} K_\nu(s\sqrt{x}) dx,$$

where $Q_0(x)$ *is defined by* (6.5).

The first version of Theorem 7.1 was established by the first author in [7, p. 311]; see also another paper by the first author [9, p. 342]. Theorem 7.1 was established via the Voronoï summation formula in [10, p. 154].

Note that both Theorems 6.1 and 7.1 can be thought of as identities involving analogues of the Hurwitz zeta function $\sum_{n=0}^{\infty}(n+a)^{-s}$, where $\sigma > 1$ and $a > 0$.

In analogy with the examples in the previous section, we provide two corresponding illustrations for Theorem 7.1. First, let $a(n) = r_k(n)$. Then $\delta = k/2$ and $\lambda_n = \mu_n = n/2$. Set $s = 2\pi\sqrt{2\beta}$, where $\mathrm{Re}(\sqrt{\beta}) > 0$. Then, for $\mathrm{Re}(\nu) > -1$,

$$\sum_{n=0}^{\infty} r_k(n) n^{(\nu+1)/2} K_{\nu+1}\left(2\pi\sqrt{n\beta}\right) = \frac{\beta^{(\nu+1)/2}\Gamma(\nu+1+k/2)}{2\pi^{k/2+\nu+1}} \sum_{n=0}^{\infty} \frac{r_k(n)}{(\beta+n)^{k/2+\nu+1}}. \tag{7.1}$$

The identity (7.1) was first proved by A. I. Popov [68, Equation (6)] in 1935. See also [15, p. 329, Corollary 4.6], where (7.1) was established by a completely different method.

Let $k = 2$ and $\nu = -1/2$ above. Revert to the notation in Theorem 7.1 by setting $s = 2\pi\sqrt{\beta}$. Then, for $\mathrm{Re}(s) > 0$,

$$\sum_{n=0}^{\infty} r_2(n) e^{-s\sqrt{n}} = 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2+4\pi^2 n)^{3/2}}, \tag{7.2}$$

in the notation of Hardy [41], who used (7.2) to establish a lower bound for the error term (analogous to (3.3)) in Gauss's *Circle Problem*. (As indicated in [41], the identity (7.2) also follows from an identity of Ramanujan.)

For another example of Theorem 7.1, set $a(n) = \tau(n)$. We therefore can immediately deduce the identity

$$\sum_{n=1}^{\infty} \tau(n) n^{(\nu+1)/2} K_{\nu+1}(s\sqrt{n}) = 2^{36+\nu} s^{\nu+1} \pi^{12} \Gamma(13+\nu) \sum_{n=1}^{\infty} \frac{\tau(n)}{(s^2+16\pi^2 n)^{\nu+13}}. \tag{7.3}$$

In particular, let $\nu = -1/2$. Then, from (7.3) and (6.7),

$$\sum_{n=1}^{\infty} \tau(n)e^{-s\sqrt{n}} = 2^{36}\pi^{23/2}\Gamma\left(\frac{25}{2}\right)\sum_{n=1}^{\infty} \frac{s\tau(n)}{(s^2 + 16\pi^2 n)^{25/2}}. \tag{7.4}$$

The identity (7.4) is originally due to Chandrasekharan and Narasimhan [23, p. 16, Equation (56)].

8. First Analogue of the Riesz Sum Identity (6.4)

In the notation (6.1), suppose that $\varphi(s)$ and $\psi(s)$ satisfy a functional equation of the form

$$\Gamma^2(\tfrac{1}{2}(s+1))\varphi(s) = \Gamma^2(\tfrac{1}{2}(2-s))\psi(1-s). \tag{8.1}$$

Set, for arbitrary ν,

$$G_\nu(x) = Y_\nu(x) + \frac{2}{\pi}e^{\pi i\nu}K_\nu(x). \tag{8.2}$$

The following theorem is proved in [9, Theorem 5.1, p. 333].

Theorem 8.1. *Let $\varphi(s)$ and $\psi(s)$ satisfy (8.1), and let ρ denote a positive integer. Then, for $\rho > 2\sigma_a^* - \frac{3}{2}$,*

$$\frac{1}{\Gamma(\rho+1)}\sideset{}{'}\sum_{\lambda_n \leq x} a(n)(x - \lambda_n)^\rho = 2^{-\rho}\sum_{n=1}^{\infty} b(n)\left(\frac{x}{\mu_n}\right)^{(\rho+1)/2} G_{\rho+1}(4\sqrt{\mu_n x}) + Q_\rho(x), \tag{8.3}$$

where $G_{\rho+1}(x)$ is defined by (8.2), where $Q_\rho(x)$ is defined by

$$Q_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(z)\psi(z)x^{z+\rho}}{\Gamma(\rho+1+z)}\,dz,$$

and where $\mathcal{C}$ is a closed curve containing the integrand's poles on its interior.

We indicate one example, which we do not explicitly record. However, readers can readily substitute the relevant parameters into (8.3). Let χ denote a non-principal, odd, primitive, character modulo k. Let $L(s,\chi)$ denote the associated Dirichlet L-function. Here,

$$\varphi(s) = \left(\frac{\pi}{k}\right)^{-(s+1)} L^2(s,\chi).$$

Since $L(s,\chi)$ is an entire function, $Q_\rho(x) \equiv 0$. Then [27, p. 71], $\lambda_n = \mu_m = \pi n/k$,

$$a(n) = \frac{k}{\pi}\sum_{rs=k} \chi(r)\chi(s) \quad \text{and} \quad b(n) = \frac{k}{\pi}\frac{i\sqrt{k}}{G(\chi)}\sum_{rs=k} \overline{\chi}(r)\overline{\chi}(s),$$

and $G(\chi)$ is the Gauss sum

$$G(\chi) = \sum_{j=1}^{k} \chi(j)e^{2\pi ij/k}.$$

9. A Second Analogue of the Riesz Sum Identity (6.4)

In the notation (6.1), suppose that $\varphi(s)$ and $\psi(s)$ satisfy a functional equation of the form

$$\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}(s-p))\varphi(s) = \Gamma(\tfrac{1}{2}(1-s))\Gamma(\tfrac{1}{2}(p+1-s))\psi(p+1-s), \tag{9.1}$$

where p is an integer. For arbitrary ν and any integer p, define

$$F_{\nu,p}(x) := \cos(\tfrac{1}{2}(p+1)\pi)J_{p+\nu}(x) - \sin(\tfrac{1}{2}(p+1)\pi)\left\{Y_{p+\nu}(x) - \frac{2}{\pi}e^{\pi i\nu}K_{p+\nu}(x)\right\}. \tag{9.2}$$

In analogy with (6.4), the following theorem holds [9, Theorem 6.1, p. 336].

Theorem 9.1. *Let $\varphi(s)$ and $\psi(s)$ satisfy (9.1). Let ρ denote a positive integer and $x > 0$. Then, for $\rho > 2\sigma_a^* - p - \frac{3}{2}$,*

$$\frac{1}{\Gamma(\rho+1)} \sum_{\lambda_n \leq x}{}' a(n)(x - \lambda_n)^\rho = 2^{-\rho} \sum_{n=1}^{\infty} b(n) \left(\frac{x}{\mu_n}\right)^{(\rho+p+1)/2} F_{\rho+1,p}(4\sqrt{\mu_n x}) + Q_\rho(x),$$

where $F_{\rho+1,p}(x)$ is defined by (9.2), $Q_\rho(x)$ is defined by

$$Q_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(z)\psi(z)x^{z+\rho}}{\Gamma(\rho+1+z)} dz,$$

and $\mathcal{C}$ is a closed curve with the integrand's poles on its interior.

As in the previous section, for brevity, we provide one example without explicitly recording the identity. Let

$$\varphi(s) := \pi^{-s}\zeta(s)\zeta(s - k), \quad k \neq 0, -1, \quad \sigma > 1, k + 1,$$

where $\zeta(s)$ denotes the Riemann zeta function. In the theorem above, let $p = k$. Then the relevant parameters are

$$\lambda_n = \mu_n = \pi n \quad \text{and} \quad a(n) = b(n) = \sigma_k(n),$$

where $\sigma_k(n)$ is defined by (6.11).

10. LOGARITHMIC SUMS

In the notation of (6.1), consider Dirichlet series $\varphi(s)$ and $\psi(s)$ satisfying a functional equation of the form

$$\Gamma^m(s)\varphi(s) = \Gamma^m(\delta - s)\psi(\delta - s),$$

where m is a positive integer and δ is a positive real number. (Note that for $m = 1$ the notation above is different from that in (6.2).) In [8], the first author established a general identity [8, pp. 362, 365] for

$$S(a; q) := \frac{1}{q!} \sum_{\lambda_n \leq x}{}' a(n) \log^q(x/\lambda_n), \tag{10.1}$$

where q is a non-negative integer, and the prime $\prime$ on the summation sign on the left-hand side indicates that if $q = 0$ and $\lambda_n = x$, then only $\frac{1}{2}a(x)$ is counted. For $q = 1$, as in (6.4), ordinary Bessel functions arise in the general identity for (10.1). We give one example. Define

$$\zeta_k(s) := \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s}, \quad \sigma > 1.$$

Then

$$\sum_{n \leq x} r_2(n) \log(x/n) = \pi x - \log x + \zeta_2'(0) - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r_2(n)}{n} J_0(2\pi\sqrt{nx}), \tag{10.2}$$

where $\zeta_2'(0)$ arises from the analytic continuation of $\zeta_2(s)$ into the entire complex plane. Oppenheim claimed that he had proved an identity for the left-hand side of (10.2), but he does not give it.

For $q = 2$, the Bessel functions $K_\nu(x)$ and $Y_\nu(x)$ appear in the general identity for (10.1). Instead of offering the general identity, we provide two special cases, namely, for the divisor function $d(n)$ and $F(n)$, the number of integral ideals of norm n in an algebraic number field. First, if γ again denotes Euler's constant,

$$\sum_{n \leq x} d(n) \log(x/n) = x(\log x - 2 + 2\gamma) + \frac{1}{4} \log(4\pi^2 x)$$

$$+ \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{d(n)}{n} \left\{ Y_0(4\pi\sqrt{nx}) + \frac{2}{\pi} K_0(4\pi\sqrt{nx}) \right\},$$

which is originally due to Oppenheim [67]; see also [8, p. 371].

Second, let h denote the class number of K, let Δ denote the discriminant of K, and let $\lambda = 2\pi R/(w|\Delta|^{1/2})$, where R is the regulator of K, and w denotes the number of roots of unity in K. Then,

$$\sum_{n \le x} F(n) \log(x/n) = \lambda h x + \zeta_K(0) \log x + \zeta_K{}'(0)$$

$$+ \frac{\sqrt{|\Delta|}}{2\pi} \sum_{n=1}^{\infty} \frac{F(n)}{n} \left\{ Y_0(4\pi\sqrt{nx/|\Delta|}) + \frac{2}{\pi} K_0(4\pi\sqrt{nx/|\Delta|}) \right\}, \quad (10.3)$$

where $\zeta_K(s)$ is the Dedekind zeta function associated with the quadratic field K, that is,

$$\zeta_K(s) := \sum_{n=1}^{\infty} \frac{F(n)}{n^s}, \quad \sigma > 1,$$

which can be analytically continued to the entire complex plane. See [8, p. 374] for a proof of (10.3).

Returning to Theorem 4.1, S. Kim and the first author [17, Theorem 5.1] proved a logarithmic analogue. First, let

$$\zeta(s, a) := \sum_{n=0}^{\infty} (n + a)^{-s}, \quad \mathrm{Re}(s) > 1, \ a > 0,$$

denote the Hurwitz zeta function. Second, the generalized Stieltjes constants, $\gamma_n(a)$, $n \ge 0$, are defined by the Laurent expansion

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n(a)(s-1)^n.$$

In particular, $\gamma_0(1) = \gamma$, Euler's constant. We now offer the aforementioned theorem from [17].

Theorem 10.1. *Let $x > 0$ and $0 < \theta < 1$. Then*

$$\sum_{n \le x} \log\left(\frac{x}{n}\right) \sum_{r|n} \cos(2\pi r\theta)$$

$$= \frac{\log(4\pi^2 x) + \gamma}{4} - x \log(2\sin(\pi\theta)) - \frac{1}{8}\left(\gamma_0(\theta) + \gamma_0(1-\theta)\right)$$

$$- \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{M_0(4\pi\sqrt{m(n+\theta)x})}{m(n+\theta)} + \frac{M_0(4\pi\sqrt{m(n+1-\theta)x})}{m(n+1-\theta)} \right\},$$

where $M_0(z)$ is defined by (3.2).

11. Theorem 7.1 An Identity Involving Hypergeometric Functions

Recall that the rising or shifted factorial $(\alpha)_n$ is defined by

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \quad n \ge 1; \qquad (\alpha)_0 = 1.$$

Let

$$_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad |z| < 1,$$

denote the ordinary hypergeometric function.

In [14], in the notation of (6.1) and (6.2), the authors proved the following theorem.

Theorem 11.1. *Assume that* $\mathrm{Re}(\nu) > -1$, $\mathrm{Re}(\sqrt{\alpha}) > \mathrm{Re}(\sqrt{\beta}) > 0$, *and* $\delta + \mathrm{Re}(\nu) + 1 > \sigma_a^* > 0$. *Let* $Q_\rho(x)$ *be defined by* (6.5), *Bessel function* $I_\nu(z)$, $K_\nu(z)$ *be defined in* (2.4), (2.5), *and suppose that* $Q_0(0)$ *exists. Assume that the integral on the right side below converges absolutely. Then,*

$$\sum_{n=1}^{\infty} a(n) I_{\nu+1}\left(\pi\sqrt{\lambda_n}\left(\sqrt{\alpha} - \sqrt{\beta}\right)\right) K_{\nu+1}\left(\pi\sqrt{\lambda_n}\left(\sqrt{\alpha} + \sqrt{\beta}\right)\right)$$

$$= \frac{2(2\pi)^{-\delta}\Gamma(\nu + \delta + 1)}{\Gamma(\nu + 2)} \sum_{n=1}^{\infty} \frac{b(n)}{\sqrt{4\mu_n + \alpha}\sqrt{4\mu_n + \beta}} \left(\frac{\sqrt{4\mu_n + \alpha} - \sqrt{4\mu_n + \beta}}{\sqrt{4\mu_n + \alpha} + \sqrt{4\mu_n + \beta}}\right)^{\nu+1}$$

$$\times \left(\frac{1}{\sqrt{4\mu_n + \alpha}} + \frac{1}{\sqrt{4\mu_n + \beta}}\right)^{2\delta-2} {}_2F_1\left[\nu - \delta + 2, 1 - \delta; \nu + 2; \left(\frac{\sqrt{4\mu_n + \alpha} - \sqrt{4\mu_n + \beta}}{\sqrt{4\mu_n + \alpha} + \sqrt{4\mu_n + \beta}}\right)^2\right]$$

$$+ \frac{Q_0(0)}{2(\nu + 1)}\left(\frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}}\right)^{\nu+1} + \int_0^\infty Q_0'(x) I_{\nu+1}\left(\pi\sqrt{x}\left(\sqrt{\alpha} - \sqrt{\beta}\right)\right) K_{\nu+1}\left(\pi\sqrt{x}\left(\sqrt{\alpha} + \sqrt{\beta}\right)\right) dx.$$

$$(11.1)$$

If we divide both sides of (11.1) by $(\sqrt{\alpha} - \sqrt{\beta})^{\nu+1}$, and let $\alpha \to \beta$, then we obtain Theorem 7.1. Details can be found in [14].

Let $a(n) = r_k(n)$ in Theorem 6.1. Then, after a considerable amount of work, we obtain the identity

$$\sum_{n=1}^{\infty} r_k(n) I_\nu\left(\pi\sqrt{n}\left(\sqrt{\alpha} - \sqrt{\beta}\right)\right) K_\nu\left(\pi\sqrt{n}\left(\sqrt{\alpha} + \sqrt{\beta}\right)\right)$$

$$= -\frac{1}{2\nu}\left(\frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}}\right)^\nu + \frac{\Gamma(k/2 + \nu)}{\pi^{k/2}2^{k-1}\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{r_k(n)}{\sqrt{n + \alpha}\sqrt{n + \beta}}\left(\frac{\sqrt{n + \alpha} - \sqrt{n + \beta}}{\sqrt{n + \alpha} + \sqrt{n + \beta}}\right)^\nu$$

$$\times \left(\frac{1}{\sqrt{n + \alpha}} + \frac{1}{\sqrt{n + \beta}}\right)^{k-2} {}_2F_1\left[1 - \frac{k}{2} + \nu, 1 - \frac{k}{2}; \nu + 1; \left(\frac{\sqrt{n + \alpha} - \sqrt{n + \beta}}{\sqrt{n + \alpha} + \sqrt{n + \beta}}\right)^2\right].$$

$$(11.2)$$

The identity (11.2) was first proved by Sun Kim and three of the present authors in [15].

As a second example, let $a(n) = \tau(n)$ in Theorem 11.1. In this case, for $\mathrm{Re}(\nu) > -13/2$,

$$\sum_{n=1}^{\infty} \tau(n) I_{\nu+1}\left(\pi\sqrt{n}\left(\sqrt{\alpha} - \sqrt{\beta}\right)\right) K_{\nu+1}\left(\pi\sqrt{n}\left(\sqrt{\alpha} + \sqrt{\beta}\right)\right)$$

$$= \frac{2(2\pi)^{-12}\Gamma(13 + \nu)}{\Gamma(\nu + 2)} \sum_{n=1}^{\infty} \frac{\tau(n)}{\sqrt{4n + \alpha}\sqrt{4n + \beta}}\left(\frac{\sqrt{4n + \alpha} - \sqrt{4n + \beta}}{\sqrt{4n + \alpha} + \sqrt{4n + \beta}}\right)^{\nu+1}$$

$$\times \left(\frac{1}{\sqrt{4n + \alpha}} + \frac{1}{\sqrt{4n + \beta}}\right)^{22} {}_2F_1\left[\nu - 10, -11; \nu + 2; \left(\frac{\sqrt{4n + \alpha} - \sqrt{4n + \beta}}{\sqrt{4n + \alpha} + \sqrt{4n + \beta}}\right)^2\right]. \quad (11.3)$$

Letting $\nu = -1/2$ in (11.3), we deduce that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{\sqrt{n}} e^{-\pi\sqrt{n}(\sqrt{\alpha} + \sqrt{\beta})} \sinh(\pi\sqrt{n}(\sqrt{\alpha} - \sqrt{\beta}))$$

$$= 2\frac{3 \cdot 5 \cdots 21}{\pi^{11}} \sum_{n=1}^{\infty} \tau(n)\left(\frac{1}{(4n + \beta)^{23/2}} - \frac{1}{(4n + \alpha)^{23/2}}\right).$$

12. Generalizations of Two Theorems of G. N. Watson

The functional equation of the Riemann zeta function $\zeta(s)$ is given by [35, p. 14]

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

Replacing s by $2s$, we see that it can be transformed into the form (6.2) with $\delta = 1/2$, $a(n) = b(n) = 1$, and $\lambda_n = \mu_n = n^2/2$.

As shown in [14, Section 15], the special case of Theorem 11.1 that arises from the Riemann zeta function is given by

$$\frac{1}{4(\nu+1)}\left(\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right)^{\nu+1} + \sum_{n=1}^{\infty} I_{\nu+1}\left(\pi n\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right) K_{\nu+1}\left(\pi n\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right)$$

$$= \frac{\Gamma(\nu+3/2)}{2\sqrt{2\pi}\Gamma(\nu+2)} \frac{\left(\sqrt{\alpha}-\sqrt{\beta}\right)^{\nu+1}}{\left(\sqrt{\alpha}+\sqrt{\beta}\right)^{\nu+2}} \cdot {}_2F_1\left(\nu+3/2, 1/2; \nu+2; \left(\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right)^2\right)$$

$$+ \frac{\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu+2)} \sum_{n=1}^{\infty} \frac{\left(\sqrt{n^2+\alpha}-\sqrt{n^2+\beta}\right)^{\nu+1}}{\left(\sqrt{n^2+\alpha}+\sqrt{n^2+\beta}\right)^{\nu+2}} \cdot {}_2F_1\left[\nu+3/2, 1/2; \nu+2; \left(\frac{\sqrt{n^2+\alpha}-\sqrt{n^2+\beta}}{\sqrt{n^2+\alpha}+\sqrt{n^2+\beta}}\right)^2\right],$$

$$(12.1)$$

where $\operatorname{Re}(\nu) > -1/2$ and $\operatorname{Re}(\sqrt{\alpha}) > \operatorname{Re}(\sqrt{\beta}) > 0$. Dividing both sides of (12.1) by $(\sqrt{\alpha}-\sqrt{b})^{\nu+1}$, letting $\alpha \to \beta$, multiplying both sides of the resulting identity by $2\Gamma(\nu+2)(2\sqrt{\beta})^{\nu+1}$, replacing ν by $\nu-1$ and β by $z^2/(4\pi^2)$, and rearranging, for $\operatorname{Re}(z) > 0$, we derive an intriguing identity of Watson [80, Equation (4)]:

$$\frac{1}{2}\Gamma(\nu) + 2\sum_{n=1}^{\infty}\left(\frac{1}{2}nz\right)^{\nu} K_{\nu}(nz) = \sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right) z^{2\nu} \left\{\frac{1}{z^{2\nu+1}} + 2\sum_{n=1}^{\infty}\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right\}.$$

Another consequence of (12.1) is another identity of Watson [80], [14, Corollary 15.2]. For $\operatorname{Re}(\beta) > 0$,

$$2\sum_{n=1}^{\infty} K_0(n\beta) = \pi\left\{\frac{1}{\beta} + 2\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{\beta^2+4\pi^2n^2}} - \frac{1}{2n\pi}\right)\right\} + \gamma + \log\left(\frac{\beta}{2}\right) - \log 2\pi.$$

13. Koshliakov's Formula and the Ramanujan–Guinand Formula

In this section, we discuss related results of N. S. Koshliakov [51], A. P. Guinand [40], and Ramanujan [71]. Guinand's Formula is probably the most famous of these theorems. However, it is equivalent to a formula discovered by Ramanujan, which can be found in a partial manuscript published with his lost notebook [71, pp. 253, 254]. We offer Ramanujan's formulation.

Theorem 13.1. (*Guinand's Formula*) [71, p. 253]. *Let $\sigma_k(n)$ be defined by (6.11), let $\zeta(s)$ denote the Riemann zeta function, and let $K_\nu(z)$ denote the modified Bessel function of order ν defined in (2.5). If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$

$$= \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2} - \alpha^{(1-s)/2}\} + \frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2} - \alpha^{(1+s)/2}\}. \quad (13.1)$$

The identity (13.1) is equivalent to an identity established by A. P. Guinand [40] in 1955, about 35–40 years after Ramanujan discovered it. Since (13.1) is known as *Guinand's Formula*, we have adhered to this convention. It is closely allied to the Fourier expansion of nonholomorphic Eisenstein series on $\mathrm{SL}(2\mathbb{Z})$ or Maass wave forms [58], and also to a famous result of Selberg and Chowla on the Epstein zeta function [24], which are discussed in Section 14.

A corollary of the Ramanujan–Guinand Formula is a famous formula of Koshliakov [51] published in 1929.

Theorem 13.2. *(**Koshliakov's Formula**) Let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if γ denotes Euler's constant and $a > 0$,*

$$\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi a n)$$

$$= \frac{1}{a}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \tag{13.2}$$

Koshliakov's proof depends upon the Voronoï summation formula (5.1). However, Koshliakov's formula (13.2) is also originally due to Ramanujan in the same aforementioned two-page manuscript. It is clear from the manuscript that Ramanujan derived Koshliakov's formula from Guinand's formula, or more appropriately, the Ramanujan-Guinand formula. Later, K. Soni [74], C. Nasim [65], and S. Yakubovich [84, p. 374] gave further proofs of (13.2). Another representation for either side of (13.2) is given by an integral containing the Riemann Ξ-function in its integrand [28, Eqeuation (1.12)], whose equivalent form can actually be found in another paper of Koshliakov [53, Equation (17)]. The identity (13.2) should be compared with (5.2).

Pages 253 and 254 of [71] contain several other beautiful formulas. We offer one of them as an example.

Theorem 13.3 (p. 254). *If $a > 0$, γ denotes Euler's constant, and $\sigma_k(n)$ is given by (6.11), then*

$$2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})$$

$$= -\frac{a^2}{2\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)} + \frac{a}{2\pi}\left((\log a + \gamma)\zeta(2) + \zeta'(2)\right) + \frac{1}{4\pi}(\log 2a\pi + \gamma) + \frac{1}{48a\pi}.$$

Theorem 13.3 was only partially stated by Ramanujan. The first author, Y. Lee, and J. Sohn [19] completed the formula and gave a proof for it. All of Ramanujan's claims on these two pages can be found in the book by G. E. Andrews and the first author [19, Chapter 3], as well as the aforementioned paper by the first author, Lee, and Sohn [19]. An equivalent formulation of Koshliakov's formula is due to the second author [28].

A generalization of the Ramanujan-Guinand formula was obtained by the second author, Kesarwani and Moll [29, Theorems 1.4, 1.5]. It contains a new generalization of the modified Bessel function $K_z(x)$, see [29, Equation (1.3)]. Several of its properties are established there. Yet another one was studied by Dixit and Kumar in [32, Equation (1.1.37)].

14. The Chowla-Selberg Formula

In 1949, S. Chowla and A. Selberg [24] offered what was later to be called the *Chowla-Selberg Formula*. However, they did not provide a proof until 18 years later [25]. To state their formula, first

let $Q(m, n) = am^2 + bmn + cn^2$ be a positive definite quadratic form with $a, b,$ and c real. The Epstein-zeta function $\mathbb{Z}(s, Q)$ is defined by

$$\mathbb{Z}(s, Q) := \sum_{m,n=-\infty}^{\infty}{}' \{Q(m, n)\}^{-s}, \quad \sigma > 1,$$

where the prime $\prime$ suggests that the term corresponding to $(m, n) = (0, 0)$ should be omitted.

Then, the *Chowla-Selberg Formula* is given by

$$a^{-s}\Gamma(s)\mathbb{Z}(s, Q) = 2\Gamma(s)\zeta(2s) + 2k^{1-2s}\Gamma(s - \tfrac{1}{2})\zeta(2s - 1)$$

$$+ 8k^{1/2-s}\pi^s \sum_{n=1}^{\infty} n^{s-1}\sigma_{1-2s}(n)\cos(n\pi b/a)K_{s-1/2}(2\pi kn). \tag{14.1}$$

Here, $\sigma_k(n)$ is defined in (6.11), $d = b^2 - 4ac$, $k = |d|/(4a^2)$, $\zeta(s)$ denotes the Riemann zeta-function, and $K_\nu(z)$ denotes the modified Bessel function, as usual.

Later proofs of (14.1) were provided by R. A. Rankin [73], P. T. Bateman and E. Grosswald [5], and Y. Motohashi [64]. The first author of the present paper established a generalization of (14.1) [11].

15. Real Analytic Eisenstein Series

Holomorphic Eisenstein series play a central role in the theory of modular forms. In particular, for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the Eisenstein series for $k > 2$ is defined by

$$E_k(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k\gamma, \tag{15.1}$$

where

$$\Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \Gamma : n \in \mathbb{N} \right\},$$

and the slash operator $|_k\gamma$ is defined as

$$(f|_k\gamma)(z) := (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right), \quad \gamma = \begin{bmatrix} a & b \\ a & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad z \in \mathbb{H}.$$

Note that we can write (15.1) as a double sum, namely,

$$E_k(z) = \frac{1}{2} \sum_{\substack{-\infty < c,d < \infty \\ (c,d)=1}} \frac{1}{(cz + d)^k}. \tag{15.2}$$

For $\sigma > 1$ and $z = x + iy \in \mathbb{H}$, a real analytic Eisenstein series on the full modular group $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$E(s, z) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s.$$

However, $E(s, z)$ is not a modular form on $\mathrm{SL}_2(\mathbb{Z})$. Similar to (15.2), $E(s, z)$ can also be expressed as a double sum

$$E(s, z) = \frac{1}{2} \sum_{\substack{-\infty < c,d < \infty \\ (c,d)=1}} \frac{y^s}{|cz + d|^{2s}}.$$

Its modular properties and Fourier expansion, which we offer below, can be employed to prove that the Riemann zeta function does not vanish on the line $\mathrm{Re}(s) = 1$, a key ingredient in one of the proofs of the prime number theorem.

For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have the modular relation [87],

$$E(s, \gamma z) = E\left(s, \frac{az + b}{cz + d}\right) = E(s, z).$$

The Fourier expansion of $E(s, z)$ is given in terms of a series involving $K_\nu(z)$, namely,

$$E(s, z) = y^s + \frac{\psi(2s - 1)}{\psi(2s)} y^{1-s} + \frac{4\sqrt{y}}{\psi(2s)} \sum_{m=1}^{\infty} m^{s - \frac{1}{2}} \sigma_{1-2s}(m) K_{s - \frac{1}{2}}(2\pi my) \cos(2\pi mx), \quad (15.3)$$

where

$$\psi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Upon using (2.10), we see that $E(s, z)$ has an analytic continuation to a meromorphic function on the entire complex plane in the variable s. Also, if $\hat{E}(s, z) = \psi(2s) E(s, z)$, then the related functional equation $\hat{E}(s, z) = \hat{E}(1 - s, z)$ holds. Indeed, note that

$$\hat{E}(s, z) = \psi(2s) y^s + \psi(2s - 1) y^{1-s} + 4\sqrt{y} \sum_{m=1}^{\infty} m^{s - \frac{1}{2}} \sigma_{1-2s}(m) K_{s - \frac{1}{2}}(2\pi my) \cos(2\pi mx).$$
$$(15.4)$$

Employing the functional equations $\psi(s) = \psi(1 - s)$, $\sigma_\nu(n) = n^\nu \sigma_{-\nu}(n)$, and $K_\nu(z) = K_{-\nu}(z)$ [81, p. 79, Equation (8)] in (15.4), we obtain the functional equation

$$\hat{E}(s, z) = \hat{E}(1 - s, z). \quad (15.5)$$

From the Fourier expansion (15.3), observe that $E(s, x + iy)$ is invariant under $x \to x + 1$. Hence, we can write

$$E(s, z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi imx},$$

where

$$a_m(y) = \begin{cases} y^s + \dfrac{\psi(2s - 1)}{\psi(2s)} y^{1-s}, & \text{if } m = 0, \\ 2\sqrt{y} |m|^{s - \frac{1}{2}} \sigma_{1-2s}(|m|) K_{s - \frac{1}{2}}(2\pi |m| y), & \text{if } m \neq 0. \end{cases}$$

We provide a brief sketch of the well-known Kronecker limit formula, which is given by

$$\frac{\partial}{\partial s} E(s, z) \Big|_{s=0} = \frac{1}{6} \log(y^6 |\Delta(z)|), \quad (15.6)$$

where $\Delta(z)$ is defined in (6.9). Recall (15.3), differentiate it with respect to s, and then employ (6.7) in the series on the right-hand side to obtain

$$\frac{\partial}{\partial s} E(s, z) \Big|_{s=0} = \log y - \frac{\pi}{3} y - \sum_{m=1}^{\infty} m^{-\frac{1}{2}} \sigma_1(m) e^{-2\pi my} \cos(2\pi mx). \quad (15.7)$$

On the other hand, if we use the definition of $\Delta(z)$, we can show that

$$\log(y^6 |\Delta(z)|) = 6 \log y + \log(|\Delta(z)|)$$

$$= 6\log(y) - 2\pi y - 24\mathrm{Re}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{nm}}{m}\right)$$

$$= 6\log(y) - 2\pi y - 24\sum_{m=1}^{\infty} m^{-\frac{1}{2}}\sigma_1(m)e^{-2\pi my}\cos(2\pi mx). \qquad (15.8)$$

Hence, from (15.7) and (15.8), we obtain (15.6).

We conclude this section by giving an application of the functional equation (15.5) in the Rankin–Selberg method. This method is very helpful in obtaining the analytic continuation of L-functions associated with modular forms. We briefly explain the method in obtaining an improved bound for Ramanujan's arithmetic function, $\tau(n)$, by employing the analytic continuation of zeta functions attached to modular forms, in particular, to the cusp form Δ-function, defined in (6.9) (see [54, Theorem 9.10]).

To begin, let us first consider

$$L(\Delta, s) := \sum_{n=1}^{\infty}\frac{\tau(n)}{n^s} = \prod_p\left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1}. \qquad (15.9)$$

Note that the $L(s, \Delta)$ is absolutely convergent for $\sigma > 13/2$. Now, if we write (15.9) as

$$L(\Delta, s) = \prod_p\left((1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})\right)^{-1},$$

where $\alpha_p + \beta_p = \tau(p)$ and $\alpha_p\beta_p = p^{11}$, then we can define,

$$L(\Delta \otimes \Delta, s) := \prod_p\left((1 - \alpha_p^2 p^{-s})(1 - \alpha_p\beta_p p^{-s})^2(1 - \beta_p^2 p^{-s})\right)^{-1}.$$

Hence, $L(\Delta \otimes \Delta, s) = \zeta(2s - 22)\sum_{n=1}^{\infty}\tau^2(n)n^{-s}$. Also, $L(\Delta \otimes \Delta, s)$ has an analytic continuation to a meromorphic function on the entire complex plane, and

$$\hat{L}(\Delta \otimes \Delta, s) := 4(2\pi)^{11-2s}\Gamma(s)\Gamma(s - 11)L(\Delta \otimes \Delta, s)$$

satisfies the functional equation $\hat{L}(\Delta \otimes \Delta, s) = \hat{L}(\Delta \otimes \Delta, s - 11)$.

In [72], Rankin applied the ideas above to $\hat{L}(\Delta \otimes \Delta, s)$, and used its simple pole at $s = 12$ to obtain the upper bound $\tau(n) = \mathcal{O}(n^{29/5})$, which improved the previous upper bound, $\tau(n) = \mathcal{O}(n^6)$. We refer readers to [54, Section 9.5] for other applications in the theory of modular forms in which K-Bessel functions make an appearance.

16. APPLICATIONS IN WHICH THE BESSEL FUNCTIONS $I_\nu(z)$ OR $Y_\nu(z)$ APPEAR

Recall that the Bessel functions of imaginary argument $I_\nu(z)$ appear in Theorem 11.1, while the Bessel functions $Y_\nu(z)$ defined in (2.2) are focused in the formulas (3.1) and (4.2). We complete our survey by mentioning two families of examples in which each of these other Bessel functions appear, unaccompanied by their siblings.

First, as seen from their asymptotic expansions (2.10) and (2.9), $K_\nu(z)$ and $I_\nu(z)$ play entirely different roles in analytic number theory. In particular, the latter functions are prominent in describing the growth of partition functions. Let $p(n)$ denote the number of partitions of the positive integer n. For example, $p(5) = 7$, because $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$. Improving on Hardy and Ramanujan's asymptotic formula for $p(n)$ [42], H. Rademacher

[70] found an exact formula for $p(n)$, namely,

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{3/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right),$$

where $I_{3/2}$ is a modified Bessel function, $A_k(n)$ is the Kloosterman sum defined by

$$A_k(n) := \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k},$$

and $s(h,k)$ denotes the Dedekind sum defined by

$$s(h,k) := \sum_{r=0}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right).$$

There are several further asymptotic formulas for partition functions in which I_ν appears, for example, in recent papers by K. Bringmann, B. Kane, L. Rolen and Z. Tripp [22] and J. Iskander, V. Jain, and V. Talvola [46].

Second, Euler's famous formula linking the Bernoulli numbers $B_{2n}, n \in \mathbb{N}$, with $\zeta(2n)$ is given by [76, p. 5, Equation (1.14)]

$$B_{2n} = \frac{2(-1)^{n+1}(2n)!\zeta(2n)}{(2\pi)^{2n}}. \tag{16.1}$$

In a beautiful paper [88], D. Zagier obtained an analogue of this formula which features the modified Bernoulli numbers

$$B_n^* := \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r}{n+r} \quad (n \in \mathbb{N}).$$

He stumbled upon B_n^* while devising a new proof of the Eichler–Selberg trace formula for the trace of Hecke operators T_ℓ acting on modular forms on $\mathrm{SL}_2(\mathbb{Z})$. The numbers B_n^* satisfy striking analogues of properties of B_n as well. Zagier's analogue of (16.1) is given by [88, Equation (10)]

$$B_{2n}^* = -n + \sum_{\ell=1}^{\infty}\left((-1)^n \pi Y_{2n}(4\pi\ell) + \frac{1}{2\sqrt{\ell}}\right) - \frac{1}{2}\zeta\left(\frac{1}{2}\right) \tag{16.2}$$

$$+ \sum_{\ell=1}^{\infty} \frac{1}{\sqrt{\ell(\ell+4)}}\left(\frac{\sqrt{\ell+4}-\sqrt{\ell}}{2}\right)^{4n},$$

where $Y_n(z)$ is defined by (2.3). The authors of [30] studied properties of the Zagier polynomials defined by

$$B_n^*(x) = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r(x)}{n+r}, \quad n > 0,$$

where $B_n^*(0) = B_n^*$. These polynomials are analogues of the Bernoulli polynomials $B_n(x)$. It is well-known that $B_{2n}(x)$ and $B_{2n+1}(x)$, respectively, have the Fourier expansions [88, p. 5]

$$B_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{m=1}^{\infty} \frac{\cos 2\pi m x}{(2\pi m)^{2n}}, \quad 0 \leq x \leq 1, n \geq 1, \tag{16.3}$$

$$B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)! \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{(2\pi m)^{2n+1}}, \tag{16.4}$$

where the last formula holds for $0 \leq x \leq 1$ when $n > 0$, and for $0 < x < 1$ when $n = 0$.

The authors of [31] studied analogues of (16.3) and (16.4) for the Zagier polynomials. We give below an analogue of (16.3) [31, Theorem 1.2].

Theorem 16.1. *Let* $0 < x < 1$ *and* $n \in \mathbb{N}$. *Define*

$$g(y, r, x) := \frac{(y + 1 + x - \sqrt{(y - 1 + x)(y + 3 + x)} \,)^{2r}}{\sqrt{(y - 1 + x)(y + 3 + x)}}.$$

Let $Y_n(z)$ *be defined in* (2.3)*, and denote by* $U_n(x)$ *the Chebyshev polynomial of the second kind, defined by*

$$U_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}.$$

Then,

$$
\begin{aligned}
B_{2n}^*(x) \;=\; & (-1)^n \pi \sum_{m=1}^{\infty} Y_{2n}(4\pi m) \cos(2\pi m x) \\
& + \frac{1}{4}\left(U_{2n-1}\left(\frac{x+1}{2}\right) + U_{2n-1}\left(\frac{x}{2}\right) + U_{2n-1}\left(\frac{x-1}{2}\right) + U_{2n-1}\left(\frac{x-2}{2}\right)\right) \\
& + \frac{1}{2^{2n+1}}\left(\sum_{m=1}^{\infty} g(m, n, x) + \sum_{m=1}^{\infty} g(m, n, 1-x)\right).
\end{aligned}
$$

There exists a similar formula for $B_{2n+1}^*(x)$ [31, Theorem 1.3]. Letting $x \to 1$ in Theorem 16.1 yields (16.2). This is analogous to obtaining (16.1) from (16.3) by substituting $x = 1$ in the latter formula.

17. APPRECIATION

REFERENCES

[1] E. Assing, *Adelic Voronoï summation and subconvexity for* GL(2) *L-functions in the depth aspect*, Int. J. Number Theory **17** (2021), 1537–1568.

[2] E. Assing and A. Corbett, *Voronoï summation via switchin cusps*, Monatsh. Math. **194** (2021), 657–685.

[3] D. Banerjee, E. M. Baruch, and D. Bump, *Voronoi summation formula for Gaussian integers*, Ramanujan J. **57** (2022), 253–274.

[4] D. Banerjee, E. M. Baruch, and E. Tenetov, *A Voronoi–Oppenheim summation formula for totally real number fields*, J. Number Theory **199** (2019), 63–97.

[5] P. T. Bateman and E. Grosswald, *On Epstein's zeta function*, Acta Arith. **9** (1964), 365-373.

[6] B. C. Berndt, *Identities Involving the Coefficients of a Class of Dirichlet Series*, Ph.D. Thesis, University of Wisconsin, 1966.

[7] B. C. Berndt, *Generalized Dirichlet series and Hecke's functional equation*, Proc. Edinburgh Math. Soc. **15** (1967), 309–313.

[8] B. C. Berndt, *Identities involving the coefficients of a class of Dirichlet series. II*, Trans. Amer. Math. Soc. **137** (1969), 361–374.

[9] B. C. Berndt, *Identities involving the coefficients of a class of Dirichlet series. III*, Trans. Amer. Math. Soc. **146** (1969), 323–348.

[10] B. C. Berndt, *Identities involving the coefficients of a class of Dirichlet series. V*, Trans. Amer. Math. Soc. **160** (1971), 139–156.

[11] B. C. Berndt, *Identities involving the coefficients of a class of Dirichlet series. VI*, Trans. Amer. Math. Soc. **160** (1971), 157–167.

[12] B. C. Berndt, *The Voronoï summation formula*, in *The Theory of Arithmetic Functions*, Kalamazoo, April, 1971, Springer–Verlag, Berlin, 1972, pp. 21–36.

[13] B. C. Berndt, A. Dixit, R. Gupta, and A. Zaharescu, *A Class of Identities Associated with Dirichlet Series Satisfying Hecke's Functional Equation*, Proc. Amer. Math. Soc. **150** (2022), 4785–4799.

[14] B. C. Berndt, A. Dixit, R. Gupta, and A. Zaharescu, *Two General Series Identities Involving Modified Bessel Functions and a Class of Arithmetical Functions*, Canad. J. Math. (2022), 1–31.

[15] B. C. Berndt, A. Dixit, S. Kim, and A. Zaharescu, *Sums of squares and products of Bessel functions*, Adv. Math. **338** (2018), 305–338.

[16] B. C. Berndt, M. Fassina, S. Kim, and A. Zaharescu, *Balanced Derivatives, Identities, and Bounds for Trigonometric and Bessel Series*, Adv. Math. **395** (2022), (40 pages).

[17] B. C. Berndt and S. Kim, *Logarithmic means and double series of Bessel functions*, Internat. J. Number Thy. **11** (2015), 1535–1556.

[18] B. C. Berndt, S. Kim, and A. Zaharescu, *The circle problem of Gauss and the divisor problem of Dirichlet–still unsolved*, Amer. Math. Monthly **125** (2018), 99–114.

[19] B. C. Berndt, Y. Lee, and J. Sohn, *Koshliakov's formula and Guinand's formula in Ramanujan's lost notebook*, in *Surveys in Number Theory*, K. Alladi, ed., Springer, New York, 2008, pp. 21–42.

[20] B. C. Berndt, J. Li, and A. Zaharescu, *The final problem: an identity from Ramanujan's lost notebook*, J. London Math. Soc. **100** (2019), 568–591.

[21] S. Bettin and J. B. Conrey, *Period functions and cotangent sums*, Algebra Number Theory **7** No. 1 (2013), 215–242.

[22] K. Bringmann, B. Kane, L. Rolen and Z. Tripp, *Fractional partitions and conjectures of Chern–Fu–Tang and Heim–Neuhauser*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 615–634.

[23] K. Chandrasekharan and R. Narasimhan, *Hecke's functional equation and arithmetical identities*, Ann. Math. **4** (1961), 1–23.

[24] S. Chowla and A. Selberg, *On Epstein's zeta-function* (I), Proc. Nat. Acad. Sci. (USA) **35** (1949), 371–374.

[25] S. Chowla and A. Selberg, *On Epstein's zeta-function*, J. Reine Angew. Math. **227** (1967), 86–110.

[26] A. Corbett, *Voronoï summation for* GL$_n$*: collusion between level and modulus*, Amer. J. Math. **143** (2021), 1361–1395.

[27] H. Davenport, *Multiplicative Number Theory*, Third ed., Springer-Verlag, New York, 2000.

[28] A. Dixit, *Series transformations and integrals involving the Riemann Ξ-function*, J. Math. Anal. Appl. **368** (2010), 358–373.

[29] A. Dixit, A. Kesarwani and V. H. Moll, *A generalized modified Bessel function and a higher level analogue of the theta transformation formula (with an appendix by N. M. Temme)*, J. Math. Anal. Appl. **459** (2018), 385–418.

[30] A. Dixit, V. H. Moll, and C. Vignat, *The Zagier modification of Bernoulli numbers and a polynomial extension. Part I*, Ramanujan J. **33**, 379–422 (2014).

[31] A. Dixit, M. L. Glasser, V. Moll and C. Vignat, *Asymptotics and exact formulas for Zagier polynomials*, Res. Number Theory, 2 (1) (2016), 26 pp.

[32] A, Dixit and R. Kumar, *Superimposing theta structure on a generalized modular relation*, Res. Math. Sci. **8** (2021) No. 3, Paper No. 41 (83 pages).

[33] A. Dixit, B. Maji and A. Vatwani, *Voronoï summation formula for the generalized divisor function* $\sigma_z^{(k)}(n)$, submitted for publication (2023).

[34] A. L. Dixon and W. L. Ferrar, *Some summations over the lattice points of a circle (I)*, Quart. J. Math. **5** (1934), 48–63.

[35] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[36] S. Egger né Endres and F. Steiner, *A new proof of the Voronoï summation formula* J. Phys. A 44 (2011), 11 pp.

[37] D. Goldfeld and X. Li, *Voronoi formulas on* GL(n). Int. Math. Res. Not. 2006, Art. ID 86295, 25 pp.

[38] D. Goldfeld and X. Li, *The Voronoi formula for* GL$(n, \mathbb{R})$, Int. Math. Res. Not. IMRN 2008, no. 2, Art. ID rnm144, 39 pp.

[39] D. Goldfeld and X. Li, Addendum to: *"The Voronoi formula for* GL$(n, \mathbb{R})$, Int. Math. Res. Not. IMRN 2008, no. 2, Art. ID rnm144, 39 pp. Int. Math. Res. Not. IMRN 2008, Art. ID rnn123, 1 p.

[40] A. P. Guinand, *Some rapidly convergent series for the Riemann ξ-function*, Quart. J. Math. (Oxford) **6** (1955), 156–160.

[41] G .H. Hardy, *On the expression of a number as the sum of two squares*, Quart. J. Pure Appl. Math. **46** (1915) 263–283.

[42] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. 2 **17** (1918), 75–115.

[43] D. A. Hejhal, *A note on the Voronoï summation formula*, Monats. Math. **87** (1979), 1–14.

[44] M. N. Huxley, *Exponential sums and lattice points. III* Proc. London Math. Soc. (3) **87** (2003), no. 3, 591-–609.

[45] A. Ichino and N. Templier, *On the Voronoï formula for* GL(n), Amer. J. Math. **135** (2013), 65-–101.

[46] J. Iskander, V. Jain, and V. Talvola, *Exact formulae for the fractional partition functions*, Res. Number Theory **6** (2020), no. 2, Paper No. 20.

[47] A. Iviv́, *On the Ternary Additive Divisor Problem and the Sixth Moment of the Zeta-Function*, in G. Greaves, G. Harman, & M. Huxley, *Sieve Methods, Exponential Sums, and their Applications in Number Theory*, London Mathematical Society Lecture Note Series, pp. 205–244, Cambridge University Press, 1997.

[48] A. Ivić, The Voronoi identity via the Laplace transform, Paul Erdös (1913–1996), Ramanujan J. **2** (1998), 39–45.

[49] M. Jutila, *A Method in the Theory of Exponential Sums*, Tata Institute Lecture Notes 80, Springer, 1987.

[50] E. M. Kıral and F. Zhou, *The Voronoi formula and double Dirichlet series*, Algebra Number Theory **10** (2016), 2267–2286.

[51] N. S. Koshliakov, *On Voronoï's sum-formula*, Mess. Math. **58** (1929), 30–32.

[52] N. S. Koshliakov, *Sur une intégrale définie et son application à la théorie des formules sommatoires*, J. Soc. Phys.-Math. Leningrad **2**, No. 2 (1929), 123–130.

[53] N. S. Koshliakov, *Some integral representations of the square of Riemann's function* $\Xi(t)$, Dokl. Akad. Nauk SSSR **2** (1934) 401–405.

[54] M. Kurokawa, M. Kurihara, and T. Saito, *Number Theory 3: Iwasawa Theory and Modular Forms*, Translations of Mathematical Monographs Iwanami Series in Modern Mathematics, **242** (2012), 226 pp.

[55] E. Landau, *Über das Konvergenzgebiet einer mit der Riemannschen Zetafunktion zusammenhängenden Reihe*, Math. Ann. **97** (1926), 251–290.

[56] A. G. F. Laurinčikas, *Voronoï and his summation formulas*. (Lithuanian) Fiz. Mat. Fak. Moksl. Semin. Darb. **2** (1999), 42–55.

[57] X. Li, *The Voronoi formula for the triple divisor function*, in *Automorphic forms and L-functions*, Int. Press, Somerville, MA, 2014, 69–90.

[58] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141–183.

[59] T. Meurman, *A simple proof of Voronoï's identity*, Asterisque **209** (1992), 265–274.

[60] S. D. Miller and W. Schmid, *Summation formulas, from Poisson and Voronoi to the present*, in *Noncommutative harmonic analysis*, Progr. Math., 220, Birkhäuser Boston, Boston, MA, 2004, pp. 419–440

[61] S. D. Miller and W. Schmid, *Automorphic distributions, L-functions, and Voronoi summation for* GL(3), Ann. Math. (2) **164** (2006), 423–488.

[62] S. D. Miller and W. Schmid, *A general Voronoi summation formula for* GL$(n, \mathbb{Z})$, in *Geometry and Analysis*, Adv. Lect. Math. (ALM), 18, Int. Press, Somerville, MA, 2011, 173–224.

[63] S. D. Miller and F. Zhou, *The balanced Voronoi formulas for* GL(n). Int. Math. Res. Not. IMRN 2019, no. 11, 3473–3484.

[64] Y. Motohashi, *A new proof of the limit formula of Kronecker*, Proc. Japan Acad. **44** (1968), 614–616.

[65] C. Nasim, *On the summation formula of Voronoi*, Trans. Amer. Math. Soc. **163** (1972), 35–45.

[66] F. Oberhettinger and K. Soni, *On some relations which are equivalent to functional equations involving the Riemann zeta function*, Math. Z. **127** (1972), 17–34.

[67] A. Oppenheim, *Some identities in the theory of numbers*, Proc. London Math. Soc. (2) **26** (1927), 295–350.

[68] A. I. Popov, *Über die zylindrische Funktionen enthaltenden Reihen* (in Russian), C. R. Acad. Sci. URSS **2** (1935), 96–99.

[69] Z. Qi, *A Voronoï–Oppenheim summation formula for number fields*, Acta Arith. **204** No. 3 (2022), 269–285.

[70] H. Rademacher, *On the partition function* $p(n)$, Proc. London Math. Soc. (2) **43** (1937), 241–254.

[71] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[72] R. A. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions*, Proc. Cambridge Philos. Soc., 35: (1939), 351–372.

[73] R. A. Rankin, *A minimum problem for the Epstein zeta-function*, Proc. Glasgow Math. Assoc. **1** (1953), 149–158.

[74] K. Soni, *Some relations associated with an extension of Koshliakov's formula*, Proc. Amer. Math. Soc. 17 (1966), 543–551.

[75] J. Steuding, *Voronoï's contribution to modern number theory*, Šiauliai Math. Semin. **2**(10) (2007), 67–106.

[76] Temme, N.M.: Special Functions. An introduction to the Classical Functions of Mathematical Physics, John Wiley and Sons, New York (1996).

[77] N. Templier, *Voronoï summation for GL(2)*, in: *Representation Theory, Automorphic Forms & Complex Geometry. A Tribute to Wilfried Schmid*, Int. Press, Somerville, MA, 2020, 163–196.

[78] G. F. Voronoï, *Sur une fonction transcendante et ses applications à la sommation de quelques séries*, Ann. École Norm. Sup. (3) **21** (1904), 207–267, 459–533.

[79] T. Watanabe, *A survey on Voronoï's theorem*, in *Geometry and analysis of automorphic forms of several variables*, World Sci. Publ., Hackensack, NJ, 2012, 334–377.

[80] G. N. Watson, *Some self-reciprocal functions*, Quart. J. Math. (Oxford) **2** (1931), 298–309.

[81] G. N. Watson, *Theory of Bessel Functions*, Cambridge University Press, second ed., London, 1966.

[82] T. A. Wong, *On the balanced Voronoï formula for* GL_N, Funct. Approx. Comment. Math. **62** (2020), 121–130.

[83] S. Yakubovich, *A general class of Voronoi's and Koshliakov-Ramanujan's summation formulas involving* $d_k(n)$, Integral Transforms Spec. Funct. **22** (2011), 801–821.

[84] S. Yakubovich, *Voronoi-Nasim summation formulas and index transforms*, Integral Transforms Spec. Funct. **23** (2012), 369–388.

[85] S. Yakubovich, *A Voronoi-type summation formula involving* $\sigma_{i\tau}(n)$ *and index transforms*, Integral Transforms Spec. Funct. **24** (2013), 98–110.

[86] V. Zacharovas, *Voronoi summation formulae and multiplicative functions on permutations*, Ramanujan J. **24** (2011), 289–329.

[87] D. Zagier, *Eisenstein series and the Riemann zeta function*, Automorphic forms, representation theory and arithmetic (Bombay, 1979) 10 (1981) 275–301.

[88] Zagier, D.: A modified Bernoulli number, Nieuw Arch. Wisk. **16** (1998), 63–72.

[89] F. Zhou, *Voronoi summation formulae on* $GL(n)$, J. Number Theory **162** (2016), 483–495.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA
   Email address: berndt@illinois.edu

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY GANDHINAGAR, PALAJ, GANDHINAGAR 382355, GUJARAT, INDIA
   Email address: adixit@iitgn.ac.in

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, 6F, ASTRONOMY-MATHEMATICS BUILDING, NO. 1, SEC. 4, ROOSEVELT ROAD, TAIPEI 106319, TAIWAN
   Email address: rajatgpt@gate.sinica.edu.tw

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA; INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, BUCHAREST RO-70700, ROMANIA
   Email address: zaharesc@illinois.edu