VORONOI SUMMATION FORMULAS, OSCILLATIONS OF RIESZ SUMS, AND RAMANUJAN-GUINAND AND COHEN TYPE IDENTITIES

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Abstract. We derive Voronoï summation formulas for the Liouville function $\lambda(n)$, the Möbius function $\mu(n)$, and for $d^2(n)$, where $d(n)$ is the divisor function. The formula for $\lambda(n)$ requires explicit evaluation of certain infinite series for which the use of the Vinogradov-Korobov zero-free region of the Riemann zeta function is indispensable. Several results of independent interest are obtained as special cases of these formulas. For example, a special case of the one for $\mu(n)$ is a famous result of Ramanujan, Hardy, and Littlewood. Cohen type and Ramanujan-Guinand type identities are established for $\lambda(n)$ and $\sigma_a(n)\sigma_b(n)$, where $\sigma_s(n)$ is the generalized divisor function. As expected, infinite series over the non-trivial zeros of $\zeta(s)$ now form an essential part of all of these formulas. A series involving $\sigma_a(n)\sigma_b(n)$ and product of modified Bessel functions occurring in one of our identities has appeared in a recent work of Dorigoni and Treilis in string theory. Lastly, we obtain results on oscillations of Riesz sums associated to $\lambda(n), \mu(n)$ and of the error term of Riesz sum of $d^2(n)$ under the assumption of the Riemann Hypothesis, simplicity of the zeros of $\zeta(s)$, the Linear Independence conjecture, and a weaker form of the Gonek-Hejhal conjecture.

Contents

1. Introduction 2
2. Main results 7
2.1. Results on the Liouville function $\lambda(n)$ 7
2.2. Results on the square of the divisor function $d^2(n)$ 9
2.3. Results on the Möbius function $\mu(n)$ 10
2.4. Cohen and Ramanujan-Guinand type identities for $\sigma_a(n)\sigma_b(n)$ 10
3. Preliminaries 13
4. Proofs of the results on $\lambda(n)$ 14
4.1. Infinite series evaluation using the Vinogradov-Korobov zero-free region 14
4.2. The Voronoi summation formula for $\lambda(n)$ 17
4.3. Corollaries of the Voronoi summation formula 21
4.4. Cohen type identity for $\lambda(n)$ 25
4.5. Ramanujan-Guinand type identity for $\lambda(n)$ 26
5. Proofs of the Voronoi summation formula for $d^2(n)$ 27
6. Proofs of the results on $\sigma_a(n)\sigma_b(n)$ 32
6.1. Cohen type identity for $\sigma_a(n)\sigma_b(n)$ 32
6.2. Ramanujan-Guinand type identity for $\sigma_a(n)\sigma_b(n)$ 35
7. Proofs of Theorem 2.9 and Ramanujan's identity (1.9) 36
8. Oscillations of Riesz sums 37
9. Concluding remarks 42
References 43

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1. Introduction

Two of the famous unsolved problems in number theory are the Gauss circle problem and the Dirichlet divisor problem. Let $d(n)$ denote the number of positive divisors of n. The Dirichlet divisor problem aims at finding the optimal error bound on $\sum_{n\leq x} d(n)$.

Almost all of the improvements on the error term of $\sum_{n\leq x} d(n)$ rely on a formula of Voronoï [63], now bearing his name, and specifically known as the *Voronoï summation formula*. For example, Voronoï himself used his result to improve the error bound from $O(\sqrt{x})$ to $O(x^{1/3}\log(x))$. A more general version of this formula, studied by Voronoï himself, involves a test function ϕ satisfying certain conditions, and is a representation for the sum $\sum_{n\leq x} d(n)\phi(n)$. Putting further restrictions on ϕ, one can even obtain a Voronoï summation formula for the infinite series $\sum_{n=1}^{\infty} d(n)\phi(n)$ in which case the formula takes the form

$$\sum_{n=1}^{\infty} d(n)\phi(n) = \int_0^{\infty}(2\gamma+\log(t))\phi(t)\mathrm{d}t + 2\pi\sum_{n=1}^{\infty} d(n)\int_0^{\infty}\phi(t)\left(\frac{2}{\pi}K_0(4\pi\sqrt{nt}) - Y_0(4\pi\sqrt{nt})\right)\mathrm{d}t, \quad (1.1)$$

where γ is Euler's constant and $Y_\nu(\xi)$ is the Bessel function of the second kind of order ν defined by [64, p. 64, Equation (1)]

$$Y_\nu(\xi) = \frac{J_\nu(\xi)\cos(\pi\nu) - J_{-\nu}(\xi)}{\sin(\pi\nu)}$$

for $\nu \notin \mathbb{Z}$, and by $Y_n(\xi) = \lim_{\nu\to n} Y_\nu(\xi)$ for $n \in \mathbb{Z}$, and where $J_\nu(\xi)$ is the Bessel function of the first kind of order ν defined by [64, p. 40]

$$J_\nu(\xi) := \sum_{m=0}^{\infty} \frac{(-1)^m(\xi/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \quad |\xi| < \infty. \quad (1.2)$$

Moreover, $K_\nu(\xi)$ is the modified Bessel function of the second kind of order ν defined by [64, p. 78, eq. (6)]

$$\begin{aligned} K_\nu(\xi) &:= \frac{\pi}{2}\frac{(I_{-\nu}(\xi) - I_\nu(\xi))}{\sin(\pi\nu)} && (\nu \in \mathbb{C}\backslash\mathbb{Z}),\\ K_n(\xi) &:= \lim_{\nu\to n} K_\nu(\xi) && (n \in \mathbb{Z}), \end{aligned} \quad (1.3)$$

where $I_\nu(\xi)$ is the modified Bessel function of the first kind is defined by

$$I_\nu(\xi) := \begin{cases} e^{-\frac{1}{2}\pi\nu i}J_\nu(e^{\frac{1}{2}\pi i}\xi), & \text{if } -\pi < \arg\xi \leq \frac{\pi}{2},\\ e^{\frac{3}{2}\pi\nu i}J_\nu(e^{-\frac{3}{2}\pi i}\xi), & \text{if } \frac{\pi}{2} < \arg\xi \leq \pi. \end{cases} \quad (1.4)$$

The reader is referred to the excellent survey articles [9], [48] on this topic.

Different authors have obtained different conditions of varying generality and applicability under which (1.1) is valid, for example, (1.1) holds if $\phi : \mathbb{R} \to \mathbb{C}$ is a smooth compactly supported function on $(0,\infty)$.

A generalization of (1.1) for the generalized divisor function $\sigma_s(n) := \sum_{d|n} d^s$ can be obtained by letting[1] $\beta \to \infty$ in [6, Theorem 6.3], thereby obtaining

$$\begin{aligned} \sum_{n=1}^{\infty}\sigma_{-s}(n)\phi(n) &= -\frac{1}{2}\phi(0+)\zeta(s) + \int_0^{\beta}(\zeta(1+s) + t^{-s}\zeta(1-s))\phi(t)\,dt\\ &\quad + 2\pi\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\int_0^{\beta} t^{-\frac{s}{2}}\phi(t)\left\{\left(\frac{2}{\pi}K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt})\right)\right.\\ &\quad \left.\times\cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\right\}dt. \end{aligned} \quad (1.5)$$

[1]The interchange of the order of limit and summation can be justified under certain extra hypotheses on ϕ than the ones given in [6, Theorem 6.3]. For example, it is definitely valid for ϕ belonging to the Schwartz class.

where $-1/2 < \mathrm{Re}(s) < 1/2$ and ϕ satisfies certain conditions, for example, see [6, Equation (6.9)]. This formula, in turn, was recently generalized in [23, Theorem 2.4] for $\sigma_z^{(k)}(n) := \sum_{d^k|n} d^z$, where $k \in \mathbb{N}$, and where ϕ is a function belonging to the Schwartz class.

Besides playing a vital role in improving the error estimates in the lattice point problems such as the Dirichlet divisor problem, the Voronoï summation formulas (1.1) and (1.5) are also useful in obtaining some important modular transformations. For example, the special case of (1.5) when $\phi(x) = e^{-xy}, \mathrm{Re}(y) > 0$, and s is an odd integer, encapsulates the modular transformations satisfied by Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ and their Eichler integrals; see [22] for more details. Also, the case $\phi(x) = e^{-xy}$ of (1.1) is used in the study of moments of the Riemann zeta function $\zeta(s)$; see [11] for the same.

A formula of the type (1.1) or (1.5) for the series $\sum_{n=1}^{\infty} a(n)\phi(n)$ (or, for that matter, for $\sum_{n\leq x} a(n)\phi(n)$), where $a(n)$ is some arithmetic function, ϕ satisfies certain hypotheses, and whose right-hand side involves an infinite series of an integral transform of ϕ is called the *Voronoï summation formula for* $a(n)$. Today such formulas are known to exist for a large class of arithmetic functions. The reader is referred to the excellent survey [42] on this topic as well as for a recent survey [7].

Among other things, this paper aims at obtaining Voronoï summation formula for an arithmetic function whose Dirichlet series evaluates to a quotient (or reciprocal) of the Riemann zeta function in a certain half-plane. It is clear that the non-trivial zeros of the zeta function ought to then play a vital role. In Theorems 2.8, and 2.1, we obtain such formulas for the square of the divisor function $d^2(n)$, the Möbius function $\mu(n)$ and the Liouville function $\lambda(n)$ respectively. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_i, 1 \leq i \leq k$ are distinct primes. Then the latter two functions are defined by

$$\mu(n) := \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } a_1 = \cdots = a_k = 1, \\ 0, & \text{else,} \end{cases}$$
$$\lambda(n) := (-1)^{a_1+a_2+\cdots+a_k}.$$

For $\mathrm{Re}(s) > 1$, the Dirichlet series of $\lambda(n), \mu(n)$ and $d^2(n)$ are respectively given by

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}, \qquad \sum_{n=1}^{\infty} \frac{\mu(n)(n)}{n^s} = \frac{1}{\zeta(s)}, \qquad \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}. \tag{1.6}$$

The formula for $d^2(n)$ respects the operation of squaring in the sense that the kernel associated to the integral transform of the test function ϕ contains two copies of $\frac{2}{\pi}K_0\big(4\sqrt{x}\big) - Y_0\big(4\sqrt{x}\big)$ in its triple integral representation; see Theorem 2.8 below. The corresponding formulas for $\lambda(n)$ and $\mu(n)$ given in Theorems 2.1 and 2.9 have simple kernels involving the sine function.

As far as the Möbius function is concerned, certain identities of such kind are known. For example. let $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ be pair of reciprocal functions in the cosine kernel, that is,

$$\tilde{\psi}(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \tilde{\phi}(u) \cos(2ux) \mathrm{d}u, \qquad \tilde{\phi}(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \tilde{\psi}(u) \cos(2ux) \mathrm{d}u. \tag{1.7}$$

Let

$$Z_1(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \tilde{\phi}(x) \mathrm{d}x, \qquad Z_2(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \tilde{\psi}(x) \mathrm{d}x \tag{1.8}$$

be the normalized Mellin transforms of $\tilde{\phi}$ and $\tilde{\psi}$ respectively. Then an identity indicated by Ramanujan to Hardy and Littlewood [30, p. 160, Equation (2.535)] reads

$$\begin{aligned} \sqrt{\alpha} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \tilde{\phi}\left(\frac{\alpha}{n}\right) - \sqrt{\beta} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \tilde{\psi}\left(\frac{\beta}{n}\right) &= \frac{1}{\sqrt{\alpha}} \sum_{\rho} \frac{\Gamma(1-\rho) Z_1(1-\rho) \alpha^{\rho}}{\zeta'(\rho)} \\ &= -\frac{1}{\sqrt{\beta}} \sum_{\rho} \frac{\Gamma(1-\rho) Z_2(1-\rho) \beta^{\rho}}{\zeta'(\rho)}, \end{aligned} \tag{1.9}$$

where $\alpha\beta = \pi$, and where ρ denotes a non-trivial zero of $\zeta(s)$. The special case $\tilde{\phi}(x) = \tilde{\psi}(x) = e^{-x^2}$, again due to Ramanujan, and discussed in [30, p. 158, Equation (2.516)], which has received a renewed attention in recent years [21], [38], [1], [16], [28] is

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi\alpha^2/n^2} - \sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi\beta^2/n^2} = \frac{1}{2\sqrt{\alpha}}\sum_{\rho}\frac{\Gamma((1-\rho)/2)}{\zeta'(\rho)}\alpha^{\rho}$$
$$= -\frac{1}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma((1-\rho)/2)}{\zeta'(\rho)}\beta^{\rho}.$$

Here, the series over the non-trivial zeros are not yet known to be convergent in the usual sense. For $m \in \mathbb{Z}$, let $\rho_m := \beta_m + i\gamma_m$ denote the m^{th} non-trivial zero of $\zeta(s)$, where $\rho_{-m} = \beta_m - i\gamma_m$. If we bracket the terms of the series in such a way that the terms for which

$$|\gamma_m - \gamma_m'| < \exp\left(-c\gamma_m/\log(\gamma_m)\right) + \exp\left(-c\gamma_m'/\log(\gamma_m')\right),$$

where $c > 0$, are included in the same bracket, then the series converges; see [61, p. 220]. It is believed that the series are not merely convergent but rather rapidly convergent and so bracketing may not be required. However, this is unproven as of now even upon the assumption of the Riemann Hypothesis (RH). The bracketing requirement can be expressed by rephrasing the series $\sum_{\rho}\frac{\Gamma((1-\rho)/2)}{\zeta'(\rho)}\alpha^{\rho}$ in the form $\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\Gamma((1-\rho_m)/2)}{\zeta'(\rho_m)}\alpha^{\rho_m}$, where, $\{T_n\}$ is a sequence tending to infinity such that $|T_n - \gamma_m| > \exp\left(-A\gamma_m/\log(\gamma_m)\right)$ for every ordinate γ_m of a zero of s [61, p. 219].

The Voronoï summation formula for $\mu(n)$ that we obtain in Theorem 2.9 gives (1.9) (and hence (2)) as special cases. This formula as well as most of our results in this paper involve infinite series over the non-trivial zeros of $\zeta(s)$ for which we use the same notation, that is, $\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}$ to indicate bracketing with $\{T_n\}$ being a certain sequence.

While both the infinite series occurring (1.1) contain the same arithmetic function $d(n)$ (and likewise, $\sigma_{-s}(n)$ in the case of (1.5)), this is not true in general. In the case of $\lambda(n)$, the corresponding function we get is the function $c(n)$ defined by

$$\sum_{n=1}^{\infty}\frac{c(n)}{n^s} := \frac{\zeta(2s-1)}{\zeta(s)} \tag{1.10}$$

where $\mathrm{Re}(s) > 1$. The series is absolutely convergent in this half-plane. It is easy to see that

$$c(n) = m\mu(k) \text{ if } n = m^2k, \text{ where } k \text{ is squarefree.} \tag{1.11}$$

Moreover, since $|\mu(k)| = \mu^2(k)$, in the same half-plane, we have

$$\sum_{n=1}^{\infty}\frac{|c(n)|}{n^s} = \sum_{m=1}^{\infty}\frac{m\cdot\mu^2(k)}{(m^2k)^s} = \zeta(2s-1)\prod_{p\text{ prime}}\left(1+p^{-s}\right) = \frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)}. \tag{1.12}$$

In the course of proving the Voronoï summation formula for $\lambda(n)$, that is, Theorem 2.1, we need to evaluate certain infinite series involving $c(n)$. For example, we prove that

$$\sum_{n=1}^{\infty}\frac{c(n)}{n} = \frac{1}{2}, \quad \sum_{n=1}^{\infty}\frac{c(n)\log n}{n} = -\frac{\gamma}{2}, \tag{1.13}$$

An interesting feature here is that one needs to make use of the Vinogradov-Korobov zero-free region for $\zeta(s)$ [62], [36] to evaluate them. The standard zero-free region is not sufficient. See Theorems 4.1 and 4.2. Note that both series in (1.13) are conditionally convergent. Several interesting corollaries of the Voronoï summation formula give interesting transformations for series involving $c(n)$ and $\lambda(n)$. See Corollaries 2.2 - 2.5.

Also, in the expression for $\sum_{n=1}^{\infty} d^2(n)\phi(n)$ that we obtain in the Voronoï summation formula for $d^2(n)$ that is, Theorem 2.8, we encounter another arithmetic function $b(n)$ which is defined by

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} := \frac{\zeta^4(s)}{\zeta(2s-1)} \qquad (\mathrm{Re}(s) > 1).$$

As shown in Lemma 5.1, $b(n)$ is multiplicative and for $n = p^k$, where p is a prime and $k \geq 0$,

$$b(p^k) = \binom{k+3}{3} - p\binom{k+1}{3}. \tag{1.14}$$

In his seminal work, Voronoï obtained the identity [63, Equation (5), (6)] (see also [55, p. 254])

$$2\sum_{n=1}^{\infty} d(n)K_0(4\pi\sqrt{nx}) = \frac{x}{\pi^2}\sum_{n=1}^{\infty} \frac{d(n)\log(x/n)}{x^2-n^2} - \frac{\gamma}{2} - \left(\frac{1}{4} + \frac{1}{4\pi^2 x}\right)\log(x) - \frac{\log(2\pi)}{2\pi^2 x}, \tag{1.15}$$

where $|\arg(x)| < \pi$. It can be derived from (1.1) by letting $\phi(t) = K_0(4\pi\sqrt{tx})$. An application of this identity is in obtaining the following identity of Koshliakov [37, Equation (5)], which is, in turn, used to derive a simpler proof of the Voronoï summation formula for $d(n)$ [37]:

$$2\sum_{n=1}^{\infty} d(n)\left(K_0\left(4\pi e^{\frac{i\pi}{4}}\sqrt{nx}\right) + K_0\left(4\pi e^{-\frac{i\pi}{4}}\sqrt{nx}\right)\right) = -\gamma - \frac{1}{2}\log x - \frac{1}{4\pi x} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{x^2+n^2}, \tag{1.16}$$

where $x \in \mathbb{C}\backslash\{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$. Soni [58] proved that both (1.15) and (1.16) are equivalent to (1.1). Another important identity associated with $d(n)$, and termed as *Koshliakov's formula* [10] (see [37]), is

$$\gamma - \log\left(\frac{4\pi}{x}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi nx) = \frac{1}{x}\left(\gamma - \log(4\pi x) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{x}\right)\right), \tag{1.17}$$

where $\mathrm{Re}(x) > 0$. Observe that the arguments of the modified Bessel function occurring in the series on the left-hand sides of (1.15) and (1.17) involve $\sqrt{x}$ and x respectively. It is known [58] that (1.17) is also equivalent to (1.1).

A generalization of (1.15) in the setting of $\sigma_{-s}(n)$ was given by Cohen [17, Theorem 3.4] who proved that for $x > 0$ and $s \notin \mathbb{Z}$, where $\mathrm{Re}(s) \geq 0$ [2], and $k \in \mathbb{Z}$ such that $k \geq \lfloor(\mathrm{Re}(s)+1)/2\rfloor$,

$$\begin{aligned} 8\pi x^{s/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_s(4\pi\sqrt{nx}) &= A(s,x)\zeta(s) + B(s,x)\zeta(s+1) \\ &+ \frac{2}{\sin(\pi s/2)}\left(\sum_{1\leq j\leq k}\zeta(2j)\zeta(2j-s)x^{2j-1} + x^{2k+1}\sum_{n=1}^{\infty}\sigma_{-s}(n)\frac{n^{s-2k}-x^{s-2k}}{n^2-x^2}\right), \end{aligned} \tag{1.18}$$

where

$$A(s,x) = \frac{x^{s-1}}{\sin(\pi s/2)} - (2\pi)^{1-s}\Gamma(s), \qquad B(s,x) = \frac{2}{x}(2\pi)^{-s-1}\Gamma(s+1) - \frac{\pi x^s}{\cos(\pi s/2)}.$$

Equation (1.16) can also be generalized in this setting; see [6, p. 844, Equation (7.3)].

Lastly, (1.17) can also be generalized to get what is known as the *Ramanujan-Guinand* formula [55, p. 253], [27], [20, Theorem 1.2], namely, for $\mathrm{Re}(x) > 0$,

$$\begin{aligned} &(\pi x)^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + (\pi x)^{s/2}\Gamma\left(\frac{-s}{2}\right)\zeta(-s) + 4\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2\pi nx) \\ &= \frac{1}{x}\left((\pi/x)^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + (\pi/x)^{s/2}\Gamma\left(\frac{-s}{2}\right)\zeta(-s) + 4\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}\left(\frac{2\pi n}{x}\right)\right). \end{aligned} \tag{1.19}$$

[2] As mentioned in [17], the condition $\sigma \geq 0$ is not restrictive since $K_{-s}(w) = K_s(w)$ implies that the left-hand side of (1.18) is invariant if we replace s by $-s$.

This has a connection with the Fourier expansion of non-holomorphic Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ and its functional equation; see [17, p. 60]. Note that Koshliakov's formula (1.17) can be viewed as the Ramanujan-Guinand formula corresponding to $d(n)$.

One may wonder how the underlying strategy for deriving the Voronoï-Ramanujan-Cohen identity (1.15) or Cohen's identity (1.18) differs from that used for proving Koshliakov's formula (1.17) or the Ramanujan-Guinand formula (1.19) as all of the identities in these formulas involve infinite series of the modified Bessel function but with different arguments. The answer is, (1.15) or (1.18) entail the asymmetric form of the functional equation of $\zeta(w)$ in their derivations, that is, [61, p. 24],

$$\Gamma(w)\zeta(w) = \frac{1}{2}(2\pi)^w\zeta(1-w)\sec\left(\frac{\pi w}{2}\right) \tag{1.20}$$

whereas, (1.17) or (1.19) require the symmetric form of the functional equation, namely,

$$\pi^{-\frac{w}{2}}\Gamma\left(\frac{w}{2}\right)\zeta(w) = \pi^{-\frac{(1-w)}{2}}\Gamma\left(\frac{1-w}{2}\right)\zeta(1-w). \tag{1.21}$$

For example, using [46, p. 115, Formula 11.1], for $c = \mathrm{Re}(z) > \pm\frac{1}{2}\mathrm{Re}(s)$,

$$\frac{1}{2\pi i}\int_{(c)}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)(4\pi^2nx)^{-z}\mathrm{d}z = 2K_s(4\pi\sqrt{nx}),$$

(where, here, and throughout the paper, by $\int_{(c)}$, we mean $\int_{c-i\infty}^{c+i\infty}$), we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_s(4\pi\sqrt{nx}) &= \frac{1}{4\pi i}\int_{(c)}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)\left(\sum_{n=1}^{\infty}\frac{\sigma_{-s}(n)n^{s/2}}{n^z}\right)(4\pi^2x)^{-z}\mathrm{d}z\\ &= \frac{1}{4\pi i}\int_{(c)}\Gamma\left(z-\frac{s}{2}\right)\Gamma\left(z+\frac{s}{2}\right)\zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right)(4\pi^2x)^{-z}\mathrm{d}z,\end{aligned}$$

provided, we now take $c = \mathrm{Re}(z) > 1 \pm \frac{1}{2}\mathrm{Re}(s)$. In the last step, we used the well-known result

$$\sum_{n=1}^{\infty}\frac{\sigma_{-s}(n)n^{s/2}}{n^z} = \zeta\left(z-\frac{s}{2}\right)\zeta\left(z+\frac{s}{2}\right).$$

The last integral can now be transformed using (1.20). But if we consider the series occurring on the left-hand side of (1.19), then performing a calculation similar to the above, for $c = \mathrm{Re}(z) > \pm\frac{1}{2}\mathrm{Re}(s)$,

$$\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2\pi nx) = \frac{1}{8\pi i}\int_{(c)}\Gamma\left(\frac{z}{2}-\frac{s}{4}\right)\zeta\left(z-\frac{s}{2}\right)\Gamma\left(\frac{z}{2}+\frac{s}{4}\right)\zeta\left(z+\frac{s}{2}\right)(\pi x)^{-z}\mathrm{d}z,$$

which needs (1.21) for simplifying the integrand.

With this mind, one may obtain new identities of these kinds by working with different arithmetic functions. Following the terminology in [3] and [4], we call identities of the type (1.15) or (1.5) as *Cohen type* identities. Also, we term those of the type (1.17) or (1.19) as *Ramanujan-Guinand type* identities.

Our second goal of this paper is to obtain Cohen type and Ramanujan-Guinand type identities, whenever possible, for functions whose Dirichlet series can be represented as quotients of Riemann zeta function. While we again concentrate on $\lambda(n)$ and $d^2(n)$, we also obtain such identities for $\sigma_a(n)\sigma_b(n)$. These can be seen in Theorems 2.6, 2.7, 2.9, 2.11, 2.13 and corollaries 2.12 and 2.14. In Remark 4, we explain why one cannot obtain such identities for $\mu(n)$.

We emphasize that there are very few studies on infinite series involving $\sigma_a(n)\sigma_b(n)$. Wigert [67] undertook a brief study of $\sum_{n=1}^{\infty}\sigma_{-1}(n)\sigma_{-\alpha}(n)e^{-nx}$, where $\alpha > 3$, and obtained a transformation for it [67, Equation (8)]. In their recent work in string theory, Dorigoni and Treilis [24, Equation (4.29)] encountered the series $\sum_{n=1}^{\infty}\sigma_a(n)\sigma_b(n)n^{-(a+b)/2}K_{a/2}(2nx)K_{b/2}(2nx)$, which is the exact series that we transform thereby obtaining the Ramanujan-Guinand identity for $\sigma_a(n)\sigma_b(n)$. Indeed, the Laplace equation considered by Dorigoni and Treilis for which the exact solution is the non-perturbative part of the

Fourier zero-mode sector has precisely this series as its inhomogeneous part [24, pp. 36-37]. This showcases the importance of such series not only in number theory but also in physics. It is our hope that other series such as the ones in (2.11) will eventually find their use in similar analyses.

We now come to the third and the last goal of our paper, namely, obtaining results on oscillations of certain weighted sums involving the arithmetic functions $\mu(n), \lambda(n)$, and of the error term corresponding to the Riesz-type sum associated to $d^2(n)$.

Historically, people have been interested in studying oscillations of functions associated to $\mu(n), \lambda(n)$ and $d(n)$, namely, $\frac{1}{\sqrt{x}}\sum_{n\leq x}\mu(n)$, $\frac{1}{\sqrt{x}}\sum_{n\leq x}\lambda(n)$ and $\frac{1}{x^{1/4}}\left(\sum_{n\leq x}d(n)-x\log(x)-(2\gamma-1)x\right)$ as $x\to\infty$. For example, Ingham [33] showed that

$$\underline{\lim}_{x\to\infty}\frac{\sum_{n\leq x}\mu(n)}{\sqrt{x}}=-\infty \quad \text{and} \quad \overline{\lim}_{x\to\infty}\frac{\sum_{n\leq x}\mu(n)}{\sqrt{x}}=\infty$$

assuming RH and the *Linear Independence Conjecture* (LI) which states that the positive imaginary ordinates of the zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$.

Using a method similar to the one that we have used for deriving our Voronoï summation formulas, we obtain oscillation results on the weighted sums of $\mu(n)$ and $\lambda(n)$ where the weights are functions of Riesz and Schwartz type. For the corresponding Riesz type sum on $d^2(n)$, we obtain a result on the oscillation of its error term. These results are stated and proved in Section 8.

2. Main results

2.1. **Results on the Liouville function $\lambda(n)$.** Let $\Phi(s)$ be holomorphic in $-1<\operatorname{Re}s<2$ except for a possible pole of order two or less at $s=0$, and such that

$$\Phi(\sigma+it)\ll t^{-1-\delta}\text{ as }t\to\pm\infty \tag{2.1}$$

for some $\delta>0$ and $-1<\sigma<2$.

Theorem 2.1. *Let $\lambda(n)$ denote the Liouville function and recall the expression for $c(n)$ from* (1.11)*. For $0<\sigma<2$, define $\phi(x):=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)x^{-s}\mathrm{d}s$, where $\Phi(s)$ satisfies* (2.1) *and the condition before* (2.1)*. For $m\in\mathbb{Z}$, let $\rho_m:=\beta_m+i\gamma_m$ denote the m^{th} non-trivial zero of $\zeta(s)$, where $\rho_{-m}=\beta_m-i\gamma_m$. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then there exists a sequence of numbers $\{T_n\}_{n=1}^{\infty}$ with $T_n\to\infty$ such that*

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda(n)\phi(n)=&\frac{1}{2\zeta(\frac12)}\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\mathrm{d}x+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\int_0^{\infty}\phi(x)x^{\rho_m-1}\mathrm{d}x\\&+\sqrt{2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}+\frac{\pi}{4}\right)\mathrm{d}x.\end{aligned} \tag{2.2}$$

Remark 1. *We allow $\Phi(s)$ to have a pole at $s=0$ in order to accommodate well-known functions ϕ belonging to Schwartz class such as e^{-x} and $K_0(x)$ as well as the functions used to construct Riesz sums.*

The condition that $\Phi(s)$ has at most a double pole at $s=0$ can, of course, be relaxed to have a pole of any order at 0, and the result should remain as above. However, the calculations for justifying this become increasingly complicated and hence we refrain from doing so.

Corollary 2.2. *Let $y>0$. Under the assumptions of Theorem* 2.1*, we have*

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda(n)e^{-ny}=&\frac{\sqrt{\pi}}{2\sqrt{y}\zeta\left(\frac12\right)}+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)\Gamma(\rho_m)}{\zeta'(\rho_m)}y^{-\rho_m}\\&+\sqrt{\pi}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\frac{\sqrt{\sqrt{4y^2+\pi^2n^2}-2y}+\sqrt{\sqrt{4y^2+\pi^2n^2}+2y}}{\sqrt{4y^2+\pi^2n^2}}.\end{aligned} \tag{2.3}$$

In addition, if we assume RH and the absolute convergence of the above series over the non-trivial zeros of $\zeta(s)$, then, as $y\to 0^+$,

$$\sum_{n=1}^{\infty}\lambda(n)e^{-ny}=O\left(\frac{1}{\sqrt{y}}\right). \tag{2.4}$$

Corollary 2.3. *Let $I_s(\xi)$ be defined in (1.4) and let $y>0$. Under the assumptions of Theorem 2.1, we have*

$$\sum_{n=1}^{\infty}\lambda(n)e^{-n^2y}=\frac{\Gamma\left(\frac{5}{4}\right)}{y^{\frac{1}{4}}\zeta\left(\frac{1}{2}\right)}+\frac{1}{2}\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\Gamma\left(\frac{\rho_m}{2}\right)y^{-\frac{\rho_m}{2}}$$
$$+\frac{\pi^{3/2}}{4\sqrt{y}}\sum_{n=1}^{\infty}c(n)e^{-\frac{\pi^2n^2}{32y}}\left(I_{-\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)+I_{\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)\right). \tag{2.5}$$

In addition, if we assume RH and the absolute convergence of the above series over the non-trivial zeros of $\zeta(s)$, then, as $y\to 0^+$,

$$\sum_{n=1}^{\infty}\lambda(n)e^{-n^2y}=O\left(\frac{1}{y^{1/4}}\right). \tag{2.6}$$

Corollary 2.4. *Let $K_s(\xi)$ be defined in (1.3) and let $y>0$. Under the assumptions of Theorem 2.1, we have*

$$\sum_{n=1}^{\infty}\lambda(n)K_0(ny)=\frac{2\sqrt{2}}{\sqrt{y}\zeta\left(\frac{1}{2}\right)}\Gamma^2\left(\frac{5}{4}\right)+\frac{1}{4}\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\Gamma^2\left(\frac{\rho_m}{2}\right)\left(\frac{2}{y}\right)^{\rho_m}$$
$$+\pi^{\frac{3}{2}}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}(\pi^2n^2+4y^2)^{\frac{1}{4}}}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{\pi n}{2\sqrt{\pi^2n^2+4y^2}}\right). \tag{2.7}$$

In addition, if we assume RH and the absolute convergence of the above series over the non-trivial zeros of $\zeta(s)$, then, as $y\to 0^+$,

$$\sum_{n=1}^{\infty}\lambda(n)K_0(ny)=O\left(\frac{1}{\sqrt{y}}\right). \tag{2.8}$$

Corollary 2.5. *Let $J_s(\xi)$ be defined in (1.2) and let $y>0$. Under the assumptions of Theorem 2.1,*

$$\sum_{n\leq y}\lambda(n)\left(1-\tfrac{n}{y}\right)^{\frac{1}{2}}$$
$$=\frac{\pi\sqrt{y}}{4\zeta\left(\frac{1}{2}\right)}+\frac{1}{2}\sqrt{\pi}\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\frac{\Gamma(\rho_m)}{\Gamma\left(\rho_m+\frac{3}{2}\right)}y^{\rho_m}$$
$$+\frac{\pi\sqrt{y}}{2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\left\{J_0\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)+\cos\left(\frac{\pi ny}{4}\right)\right)+J_1\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)-\cos\left(\frac{\pi ny}{4}\right)\right)\right\}. \tag{2.9}$$

In addition, if we assume RH and the absolute convergence of the above series over the non-trivial zeros of $\zeta(s)$, then, as $y\to\infty$,

$$\sum_{n\leq y}\lambda(n)\left(1-\tfrac{n}{y}\right)^{\frac{1}{2}}=O\left(\sqrt{y}\right). \tag{2.10}$$

Remark 2. *There exists a transformation for the sum* $\sum_{n\leq y}\lambda(n)\left(1-n/y\right)^{k}$, *where* $k>-1$, *which generalizes the above result as the integral inside the series on the right-hand side of* (2.2) *can be explicitly evaluated. However, the evaluation in terms of generalized hypregeometric functions is complicated as opposed to the elegant Bessel function evaluation in* (2.9). *Hence we refrain from giving the same.*

We now obtain Cohen-type identity associated with $\lambda(n)$.

Theorem 2.6. *Let* $x>0$ *and let* $c(n)$ *be given in* (1.11). *For* $m\in\mathbb{Z}$, *let* $\rho_m:=\beta_m+i\gamma_m$ *denote the* m^{th} *non-trivial zero of* $\zeta(s)$, *where* $\rho_{-m}=\beta_m-i\gamma_m$. *Assume that the non-trivial zeros of* $\zeta(s)$ *are simple. Then there exists a sequence of numbers* $\{T_n\}_{n=1}^{\infty}$ *with* $T_n\to\infty$ *such that*

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n(x^2+n^2)}=-\pi x^{-5/2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{2n}}e^{-\pi nx/2}-\frac{\pi x^{-5/2}}{2\sqrt{2}\zeta\left(\frac{1}{2}\right)}+\frac{2\pi^2}{x^2}\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m-1)\Gamma(2\rho_m-1)}{\zeta'(\rho_m)\Gamma(\rho_m)(2\pi x)^{\rho_m}}. \tag{2.11}$$

Remark 3. *This identity should be compared with* (1.16).

We conclude this subsection by stating the Ramanujan-Guinand type identity associated with $\lambda(n)$ and $c(n)$.

Theorem 2.7. *There exists a sequence of numbers* $\{T_n\}_{n=1}^{\infty}$ *with* $T_n\to\infty$ *such that*

$$\begin{aligned}\frac{x}{2}\sum_{n=1}^{\infty}n\lambda(n)e^{-\pi n^2x^2/4}&=\frac{1}{4\sqrt{2}x^2}\sum_{n=1}^{\infty}nc(n)e^{-\pi n^2/(8x^2)}\left(K_{1/4}\left(\frac{\pi n^2}{8x^2}\right)+K_{3/4}\left(\frac{\pi n^2}{8x^2}\right)\right)+\frac{\pi^{1/4}}{2\sqrt{x}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)}\\&\quad+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\frac{\Gamma(\rho_m)}{\Gamma\left(\rho_m/2\right)}(\sqrt{\pi}x)^{-\rho_m}.\end{aligned} \tag{2.12}$$

2.2. **Results on the square of the divisor function** $d^2(n)$**.** Let $\Phi(s)$ be a holomorphic function in the strip[3] $-1<\operatorname{Re}s<2$ and such that

$$\Phi(\sigma+it)\ll t^{-3-\delta} \tag{2.13}$$

as $t\to\infty$ for some $\delta>0$ and $-1<\sigma<2$.

Theorem 2.8. *Let* $d(n)$ *be the divisor function and recall the expression for* $b(n)$ *from* (1.14). *For* $-1<\sigma<2$, *define* $\phi(x):=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)x^{-s}\mathrm{d}s$, *where* $\Phi(s)$ *satisfies the aforementioned conditions. For* $m\in\mathbb{Z}$, *let* $\rho_m:=\beta_m+i\gamma_m$ *denote the* m^{th} *non-trivial zero of* $\zeta(s)$, *where* $\rho_{-m}=\beta_m-i\gamma_m$. *Assume that the non-trivial zeros of* $\zeta(s)$ *are simple. Then there exists a sequence of numbers* $\{T_n\}_{n=1}^{\infty}$ *with* $T_n\to\infty$ *such that*

$$\begin{aligned}\sum_{n=1}^{\infty}d^2(n)\phi(n)&=\int_0^{\infty}(A_0+A_1\log(x)+A_2\log^2(x)+A_3\log^3(x))\phi(x)\mathrm{d}x\\&\quad+\lim_{T_n\to\infty}\sum_{|\gamma_m|\leq T_n}\frac{\zeta^4(\frac{\rho_m}{2})}{2\zeta'(\rho_m)}\int_0^{\infty}\phi(x)x^{\frac{\rho_m}{2}-1}dx\\&\quad+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{b(n)}{n}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left(\frac{2}{\pi}K_0\big(4\sqrt{x}\big)-Y_0\big(4\sqrt{x}\big)\right)\left(\frac{2}{\pi}K_0\big(4\sqrt{y}\big)-Y_0\big(4\sqrt{y}\big)\right)\\&\qquad\times\frac{\phi(z)}{z}\cos\left(\frac{2\sqrt{xy}}{\pi\sqrt{nz}}\right)\mathrm{d}z\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{2.14}$$

[3] Unlike in Theorem 2.1, we do not allow the function $\Phi(s)$ in Theorem 2.8 to have a pole at $s=0$ so as to not compromise on the elegance of the result, but, in principle, this can be done.

where

$$\begin{aligned}A_0 &= \frac{1}{\pi^8}\big(24\gamma^3\pi^6 - 72\gamma\pi^6\gamma_1 + 12\pi^6\gamma_2 - 432\gamma^2\pi^4\zeta'(2) + 288\pi^4\gamma_1\zeta'(2) + 3456\gamma\pi^2\zeta'(2)^2 - 10368\zeta'(2)^3\\ &\quad - 288\gamma\pi^4\zeta''(2) + 1728\pi^2\zeta'(2)\zeta''(2) - 48\pi^4\zeta'''(2)\big),\\ A_1 &= \frac{1}{\pi^8}\left(36\gamma^2\pi^6 - 24\pi^6\gamma_1 - 288\gamma\pi^4\zeta'(2) + 864\pi^2\zeta'(2)^2 - 72\pi^4\zeta''(2)\right),\\ A_2 &= \frac{1}{\pi^8}\left(12\gamma\pi^6 - 36\pi^4\zeta'(2)\right),\\ A_3 &= \frac{1}{\pi^2}, \end{aligned}\tag{2.15}$$

with γ_1 being the first Stieltjes constant.

2.3. **Results on the Möbius function $\mu(n)$.** Let $\Phi(s)$ be holomorphic in $-1 < \mathrm{Re}\, s < 2$ except for a possible simple pole at $s = 0$, and such that

$$\Phi(\sigma + it) \ll t^{-1-\delta}\tag{2.16}$$

as $t \to \pm\infty$ for some $\delta > 0$ and $-1 < \delta < 2$.

Theorem 2.9. *For $0 < \sigma < 2$, define $\phi(x) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(s)x^{-s}\mathrm{d}s$, where $\Phi(s)$ satisfies the aforementioned conditions. For $m \in \mathbb{Z}$, let $\rho_m := \beta_m + i\gamma_m$ denote the m^{th} non-trivial zero of $\zeta(s)$, where $\rho_{-m} = \beta_m - i\gamma_m$. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then there exists a sequence of numbers $\{T_n\}_{n=1}^{\infty}$ with $T_n \to \infty$ such that*

$$\sum_{n=1}^{\infty}\mu(n)\phi(n) = -2k + \lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{1}{\zeta'(\rho_m)}\int_0^{\infty}\phi(x)x^{\rho_m-1}\mathrm{d}x - 4\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\int_0^{\infty}(\phi(t)-k)\left(\frac{\sin^2(\pi/nt)}{t}\right)\mathrm{d}t,$$

where k is the residue of $\Phi(s)$ at $s = 0$.

In Section 7, we briefly sketch the proof of the above theorem since it is similar to that of Theorem 2.1, and since the latter is proved in complete detail in Section 4.2.

Corollary 2.10. *Ramanujan's identity* (1.9) *holds.*

Remark 4. *One does not get Ramanujan-Guinand type and Cohen-type identities for $\mu(n)$ since the corresponding integrals $\frac{1}{2\pi i}\int_{(c)}\frac{x^{-s}\mathrm{d}s}{\pi^{-s/2}\Gamma(s/2)\zeta(s)}$ and $\frac{1}{2\pi i}\int_{(d)}\frac{x^{-s}\mathrm{d}s}{\Gamma(s)\zeta(s)}$ diverge along any vertical line.*

2.4. **Cohen and Ramanujan-Guinand type identities for $\sigma_a(n)\sigma_b(n)$.** Although we do not venture[4] into deriving the Voronoï summation formula for $\sigma_a(n)\sigma_b(n)$, we do obtain Cohen-type identity as well the analogue of Ramanujan-Guinand identity for this arithmetic function.

Before stating the Cohen-type identity for $\sigma_a(n)\sigma_b(n)$, we need to define an arithmetic function $C_{a,b}(n)$ occurring in our formulas below. It is defined by means of the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{C_{a,b}(n)}{n^s} := \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b-1)} \qquad (\mathrm{Re}(s) > \eta),\tag{2.17}$$

where

$$\eta := \max\{1, 1+\mathrm{Re}(a), 1+\mathrm{Re}(b), 1+\mathrm{Re}(a+b), 1+\mathrm{Re}(a+b)/2\}.\tag{2.18}$$

We now give an expression for $C_{a,b}(n)$ as the Dirichlet convolution of two familiar functions. Write the right-hand side of (2.17) as

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b-1)} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\cdot\frac{\zeta(2s-a-b)}{\zeta(2s-a-b-1)}.\tag{2.19}$$

[4]In principle, this can, of course, be done, and a formula generalizing that for $d^2(n)$, that is, (2.14) can be derived; however, it is quite complicated and hence we refrain from going in that direction.

Now for $\mathrm{Re}(s) > \eta$, we have [61, p. 8, Equation (1.3.3)],

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}. \tag{2.20}$$

Morever, if ϕ^{-1} denotes the Dirichlet inverse of the Euler totient function ϕ, then, for $\mathrm{Re}(s) > 2$, we have $\sum_{n=1}^{\infty} \phi^{-1}(n)n^{-s} = \zeta(s)/\zeta(s-1)$, whence, for $\mathrm{Re}(s) > \eta$,

$$\frac{\zeta(2s-a-b)}{\zeta(2s-a-b-1)} = \sum_{n=1}^{\infty} \frac{\kappa_{a,b}(n)}{n^s},$$

where

$$\kappa_{a,b}(n) = \begin{cases} m^{a+b}\phi^{-1}(m), \text{if } n = m^2, \\ 0, \qquad\qquad \text{else.} \end{cases} \tag{2.21}$$

Thus, from (2.19)-(2.21), we see that

$$C_{a,b}(n) = \left(\sigma_a\sigma_b * \kappa_{a,b}\right)(n). \tag{2.22}$$

Theorem 2.11. *Let $-1 < \mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(a-b), \mathrm{Re}(a+b) < 1$ and $x > 0$. Let $C_{a,b}(n)$ be given in (2.22). Let $f(s) := \Gamma(s)\zeta(s)$. For $m \in \mathbb{Z}$, let $\rho_m := \beta_m + i\gamma_m$ denote the m^{th} non-trivial zero of $\zeta(s)$, where $\rho_{-m} = \beta_m - i\gamma_m$. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then there exists a sequence of numbers $\{T_n\}_{n=1}^{\infty}$ with $T_n \to \infty$ such that*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n) \frac{x\left(x^{-a} - n^{-a}\right)\left(x^{-b} - n^{-b}\right)}{x^2 - n^2} \\
&= 32\pi x^{\frac{1-a-b}{2}} \sin\left(\frac{\pi a}{2}\right) \sin\left(\frac{\pi b}{2}\right) \sum_{n=1}^{\infty} \frac{C_{a,b}(n)}{n^{\frac{a+b-1}{2}}} \bigg(K_{a-1}\Big(4\pi\sqrt{nx}\Big) K_b\Big(4\pi\sqrt{nx}\Big) \\
&\quad + K_{b-1}\Big(4\pi\sqrt{nx}\Big) K_a\Big(4\pi\sqrt{nx}\Big) + \frac{(a+b-1)}{4\pi\sqrt{nx}} K_b\Big(4\pi\sqrt{nx}\Big) K_a\Big(4\pi\sqrt{nx}\Big) \bigg) - 2(2\pi)^{a+b} \\
&\quad \times \sin\left(\frac{\pi a}{2}\right) \sin\left(\frac{\pi b}{2}\right) \left(\sum_{k=0}^{1} \left(R_k(x) + R_{k+a}(x) + R_{k+b}(x) + R_{k+a+b}(x)\right) + \lim_{T_n \to \infty} \sum_{|\gamma_m| < T_n} R_{\rho_m,a,b}(x) \right),
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
R_0(x) &= \frac{-f(-a)f(-b)f(-a-b)}{2f(-a-b-1)}, & R_a(x) &= \frac{-f(a)f(a-b)f(-b)}{2f(a-b-1)}(4\pi^2x)^{-a}, \\
R_b(x) &= \frac{-f(b)f(b-a)f(-a)}{2f(b-a-1)}(4\pi^2x)^{-b}, & R_{a+b}(x) &= \frac{-f(a+b)f(a)f(b)}{2f(a+b-1)}(4\pi^2x)^{-b-a}, \\
R_1(x) &= f(1-a)f(1-b)(4\pi^2x)^{-1}, & R_{1+a}(x) &= f(1+a)f(1-b)(4\pi^2x)^{-1-a}, \\
R_{1+b}(x) &= f(1+b)f(1-a)(4\pi^2x)^{-1-b}, & R_{1+a+b}(x) &= f(1+a)f(1+b)(4\pi^2x)^{-1-a-b},
\end{aligned} \tag{2.24}$$

and

$$R_{\rho_m,a,b}(x) = \frac{f((1+\rho_m+a+b)/2)f((1+\rho_m-a+b)/2)f((1+\rho_m+a-b)/2)f((1+\rho_m-a-b)/2)}{2\zeta'(\rho_m)\Gamma(\rho_m)(4\pi^2x)^{\frac{\rho_m+1+a+b}{2}}}. \tag{2.25}$$

As a corollary, we get the Cohen-type identity for $d^2(n)$.

Corollary 2.12. *Let $x > 0$. Recall the notation for the non-trivial zeros of $\zeta(s)$ from Theorem 2.11 and assume that they are simple. Let $C_{a,b}(n)$ be defined in (2.17), or equivalently, by (2.22). Then there exists a sequence of numbers $\{T_n\}_{n=1}^{\infty}$ with $T_n \to \infty$ such that*

$$\sum_{n=1}^{\infty} d^2(n)\frac{x\log^2(x/n)}{x^2-n^2} = 8\pi^3\sqrt{x}\sum_{n=1}^{\infty}\sqrt{n}C_{0,0}(n)\left(2K_0(4\pi\sqrt{nx})K_1(4\pi\sqrt{nx}) - \frac{K_0^2(4\pi\sqrt{nx})}{4\pi\sqrt{nx}}\right) - \frac{\pi^2}{2}\left(4R_0(x)+4R_1(x)+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n} R_{\rho_m,0,0}(x)\right), \tag{2.26}$$

where

$$R_{\rho_m,0,0}(x) = \frac{\Gamma^4\left(\frac{1+\rho_m}{2}\right)\zeta^4\left(\frac{1+\rho_m}{2}\right)}{2\zeta'(\rho_m)\Gamma(\rho_m)(4\pi^2x)^{\frac{\rho_m+1}{2}}}, \qquad R_1(x) = \frac{\log^2(4\pi^2x)}{4\pi^2x},$$

and

$$R_0(x) = \frac{\pi^2}{4} + 3\log^2(2) - 72\log(A) + 6(\gamma+12\log(A))(12\log(A)-\log(2\pi)) + \log(\pi)\log(64\pi^3) + \frac{3}{4}\log(x)\left(4\gamma+48\log(A)-4\log(2\pi)+\log(x)\right) - 6\gamma_1 + 36\zeta''(-1),$$

with A being the Glaisher-Kinkelin constant.

Remark 5. *The above identity should be compared with (1.15). Also, for $a = b = 0$, the arithmetic function $C_{a,b}(n)$ can be written in an alternative form, namely, $C_{0,0}(n) = b(n)$, where $b(1) = 1$, and for primes p, $b(p^k)$ is defined in (1.14).*

We now state the Ramanujan-Guinand identity for $\sigma_a(n)\sigma_b(n)$.

Theorem 2.13. *Let $a, b \in \mathbb{C}$ and $x > 0$. Let $C_{a,b}(n)$ be defined by (2.17). Let $g(s) := \Gamma(s/2)\zeta(s)$. For $m \in \mathbb{Z}$, let $\rho_m := \beta_m + i\gamma_m$ denote the m^{th} non-trivial zero of $\zeta(s)$, where $\rho_{-m} = \beta_m - i\gamma_m$. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then there exists a sequence of numbers $\{T_n\}_{n=1}^{\infty}$ with $T_n \to \infty$ such that*

$$\frac{8}{x^{\frac{a+b}{2}}}\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^{(a+b)/2}}K_{\frac{a}{2}}(2nx)K_{\frac{b}{2}}(2nx) = \frac{2^{\frac{3-a-b}{2}}}{\pi^{a+b+1}}\sum_{n=1}^{\infty}\frac{C_{-a,-b}(n)}{n}G_{4,\,2}^{0,\,4}\left(\begin{matrix}\frac{1}{2}, \frac{1-a}{2}, \frac{1-b}{2}, \frac{1-a-b}{2}\\ \frac{1-a-b}{4}, \frac{3-a-b}{4}\end{matrix}\,\middle|\,\frac{x^2}{4n^2\pi^4}\right) + \sum_{k=0}^{1}\left(R_k(x)+R_{k+a}(x)+R_{k+b}(x)+R_{k+a+b}(x)\right) + \lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n} R_{\frac{\rho_m+a+b}{2}}(x), \tag{2.27}$$

where $G_{4,\,2}^{0,\,4}$ is the Meijer G-function defined in (3.3), and

$$R_0(x) = -g(-a)g(-b), \qquad R_a(x) = -g(a)g(-b)x^{-a},$$
$$R_b(x) = -g(b)g(-a)x^{-b} \qquad R_{a+b}(x) = -g(a)g(b)x^{-a-b},$$
$$R_1(x) = \frac{\sqrt{\pi}g(1-a)g(1-b)g(1-a-b)}{g(2-a-b)x}, \qquad R_{1+a}(x) = \frac{\sqrt{\pi}g(1+a)g(1+a-b)g(1-b)}{g(2+a-b)x^{1+a}},$$
$$R_{1+b}(x) = \frac{\sqrt{\pi}g(1+b)g(1+b-a)g(1-a)}{g(2+b-a)x^{1+b}}, \qquad R_{1+a+b}(x) = \frac{\sqrt{\pi}g(1+a+b)g(1+a)g(1+b)}{g(2+a+b)x^{1+a+b}}, \tag{2.28}$$

and

$$R_{\frac{\rho_m+a+b}{2}}(x)=\frac{1}{2\Gamma\left(\frac{\rho_m}{2}\right)\zeta'(\rho_m)x^{\frac{\rho_m+a+b}{2}}}g\left(\frac{\rho_m+a+b}{2}\right)g\left(\frac{\rho_m-a+b}{2}\right)g\left(\frac{\rho_m+a-b}{2}\right)g\left(\frac{\rho_m-a-b}{2}\right). \tag{2.29}$$

Remark 6. *As mentioned in the introduction, the series on the left-hand side of* (2.27) *has recently turned up in the work of Dorigoni and Treilis* [24] *in string theory.*

The following corollary gives the corresponding Ramanujan-Guinand identity for $d^2(n)$.

Corollary 2.14. *Under the same hypotheses in Theorem* 2.13*, we have*

$$4\left(\gamma-\log\left(\tfrac{4\pi^2}{x}\right)\right)^2+8\sum_{n=1}^{\infty}d^2(n)K_0^2(2nx)$$
$$=\frac{2^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{C_{0,0}(n)}{n}G^{0,\,4}_{4,\,2}\left(\begin{matrix}\frac12,\frac12,\frac12,\frac12\\ \frac14,\frac34\end{matrix}\middle|\,\frac{x^2}{4n^2\pi^4}\right)+4\tilde{R}_1(x)+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\Gamma^4(\rho_m/4)\zeta^4(\rho_m/2)}{2\Gamma\left(\frac{\rho_m}{2}\right)\zeta'(\rho_m)}x^{-\rho_m/2},$$

where $x\tilde{R}_1(x)$ is a complicated cubic polynomial in $\log(x)$*.*

Remark 7. *The left-hand side of Corollary* 2.14 *should be compared with that of* (1.17)*, the corresponding formula for $d(n)$.*

Remark 8. *While the Meijer G-function occurring in Theorem* 2.13 *does not admit a representation in terms of well-known functions, it can be represented as a double integral involving the modified Bessel function of the second kind. Indeed, using the basic properties of the Meijer G-function, namely,*

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}G^{m,\,n}_{p,\,q}\left(\begin{matrix}a_1,a_2,\cdots,a_p\\ b_1,b_2,\cdots,b_q\end{matrix}\middle|\,zx\right)\mathrm{d}x=\Gamma(\beta)G^{m,\,n+1}_{p+1,\,q+1}\left(\begin{matrix}1-\alpha,a_1,a_2,\cdots,a_p\\ b_1,b_2,\cdots,b_q,1-\alpha-\beta\end{matrix}\middle|\,z\right),$$

and [51, p. 621, Formula (39)]*, namely, for $m\leq q-1$,*

$$\frac{d}{dz}\left[z^{-b_q}G^{m,\,n}_{p,\,q}\left(\begin{matrix}a_1,\cdots,a_n,a_{n+1},\cdots,a_p\\ b_1,\cdots,b_m,b_{m+1},\cdots,b_q\end{matrix}\middle|\,z\right)\right]=z^{-b_q-1}G^{m,\,n}_{p,\,q}\left(\begin{matrix}a_1,\cdots,a_n,a_{n+1},\cdots,a_p\\ b_1,\cdots,b_m,b_{m+1},\cdots,b_{q-1},b_q+1\end{matrix}\middle|\,z\right),$$

it is not difficult to see that

$$G^{0,\,4}_{4,\,2}\left(\begin{matrix}\frac12,\frac{1-a}{2},\frac{1-b}{2},\frac{1-a-b}{2}\\ \frac{1-a-b}{4},\frac{3-a-b}{4}\end{matrix}\middle|\,y\right)=\sqrt{y}G^{0,\,4}_{4,\,2}\left(\begin{matrix}0,\frac{-a}{2},\frac{-b}{2},\frac{-a-b}{2}\\ \frac{-a-b-1}{4},\frac{1-a-b}{4}\end{matrix}\middle|\,y\right)$$
$$=\frac{2\sin\left(\pi\left(\frac{1-a-b}{4}\right)\right)}{\pi y^{(a+b+1)/2}}\int_0^1\int_0^1\frac{(1-x)^{\frac{-3-a-b}{4}}(1-t)^{\frac{a+b-1}{4}}}{xt^{1+\frac{a+b}{2}}}\left\{\tfrac{(2a-1)}{4}K_{\frac{a-b}{2}}\left(\tfrac{2}{\sqrt{txy}}\right)+\tfrac{1}{\sqrt{txy}}K_{1+\frac{a-b}{2}}\left(\tfrac{2}{\sqrt{txy}}\right)\right\}\mathrm{d}x\mathrm{d}t.$$

3. Preliminaries

Stirling's formula for the Gamma function in the vertical strip $p\leq\sigma\leq q$ is given by [18, p. 224]

$$|\Gamma(\sigma+it)|=\sqrt{2\pi}|t|^{\sigma-\frac12}e^{-\frac12\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right)\qquad(|t|\to\infty) \tag{3.1}$$

Theorem 3.1 (Parseval's formula)**.** [47, p. 83, Equation (3.1.13)] *Let $F(s)$ and $G(s)$ be the Mellin transforms of $f(x)$ and $g(x)$ respectively. If $F(1-s)$ and $G(s)$ have a common strip of analyticity, then for any vertical line* $\mathrm{Re}(s)=c$ *in the common strip, we have*

$$\frac{1}{2\pi i}\int_{(c)}G(s)F(1-s)ds=\int_0^\infty f(t)g(t)dt, \tag{3.2}$$

under the assumption that the integral on the right-hand side exists and the conditions

$$t^{c-1}g(t) \in L[0,\infty) \quad \text{and} \quad F(1-c-it) \in L(-\infty,\infty)$$

hold.

We now define the Meijer G-function [51, p. 617, Definition **8.2.1**]. Let m, n, p, q be integers such that $0 \leq m \leq q$, $0 \leq n \leq p$. Let $a_1, \cdots, a_p, b_1, \cdots, b_q \in \mathbb{C}$ such that $a_i - b_j \notin \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Then the Meijer G-function is defined by

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots b_m, b_{m+1}, \cdots, b_q \end{matrix} \middle| z \right) := \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j + w) \prod_{j=1}^n \Gamma(1 - a_j - w) z^{-w}}{\prod_{j=m+1}^q \Gamma(1 - b_j - w) \prod_{j=n+1}^p \Gamma(a_j + w)} \, dw, \tag{3.3}$$

where L goes from $-i\infty$ to $+i\infty$ and separates the poles of $\Gamma(1 - a_j - w)$ from the poles of $\Gamma(b_j + w)$. The integral converges absolutely if $p + q < 2(m + n)$ and $|\arg(z)| < \left(m + n - \frac{p+q}{2}\right)\pi$. In the case $p + q = 2(m + n)$ and $\arg(z) = 0$, the integral converges absolutely if $\left(\mathrm{Re}(w) + \frac{1}{2}\right)(q - p) > \mathrm{Re}(\psi) + 1$, where $\psi = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j$.

4. Proofs of the results on $\lambda(n)$

Before proving Theorem 2.1, we derive some crucial lemmas in the following subsection. These lemmas are interesting in themselves.

4.1. **Infinite series evaluation using the Vinogradov-Korobov zero-free region.**

Recall the definition of $c(n)$ from (1.10). In the Voronoï summation formula for $\lambda(n)$ given in Theorem 2.1, we crucially require the exact evaluation of the Dirichlet series of $c(n)$ at $s = 1$. If we let $s \to 1$ on both sides of (1.10) and *formally* interchange the order of limit and summation on the left-hand side, we do get a correct evaluation. However, the rigorous justification of this result is delicate and requires the use of the Vinogradov-Korobov zero-free region. This is exactly what is done next.

Theorem 4.1. *Let $c(n)$ be given by* (1.11). *We have*

$$\sum_{n=1}^{\infty} \frac{c(n)}{n} = \frac{1}{2}. \tag{4.1}$$

Proof. From (1.10), we have

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{1+w}} = \frac{\zeta(1+2w)}{\zeta(1+w)}$$

for $\mathrm{Re}\, w > 0$. From [43, p. 140, Corollary 5.3], for T large enough, $1 < x < T$, and $\sigma_0 > 0$ (to be chosen later),

$$\sum_{n \leq x} \frac{c(n)}{n} = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w} \mathrm{d}w + R \tag{4.2}$$

where

$$R \ll \sum_{\substack{x/2 < n < 2x \\ n \neq x}} \frac{|c(n)|}{n} \min\left(1, \frac{x}{T|x-n|}\right) + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|c(n)|}{n^{1+\sigma_0}}.$$

Now we first approximate the error term R. Using (1.12), we see that

$$\sum_{n=1}^{\infty} \frac{|c(n)|}{n^{1+\sigma_0}} = \frac{\zeta(1+\sigma_0)\zeta(1+2\sigma_0)}{\zeta(2+2\sigma_0)} \ll \frac{1}{\sigma_0^2}$$

as $\sigma_0 \to 0^+$. As $c(n) \ll \sqrt{n}$,

$$\sum_{\substack{x/2 < n < 2x \\ n \neq x}} \frac{|c(n)|}{n} \min\left(1, \frac{x}{T|x-n|}\right) \ll \frac{\sqrt{x} \log x}{T}. \tag{4.3}$$

Thus,

$$R \ll \frac{x^{\sigma_0}}{T\sigma_0^2} + \frac{\sqrt{x}\log x}{T}.$$

We take $\sigma_0 = \frac{1}{\log x}$ to get

$$R \ll \frac{\sqrt{x}\log x}{T}.$$

Now we will compute

$$I = \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w} \mathrm{d}w.$$

From Vinogradov-Korobov zero-free region [36], [62], we know that $\zeta(1+w)$ has no zeros when $\sigma \geq -\frac{c}{\log^{2/3}(t)(\log\log(t))^{1/3}}$ for some $c > 0$. (For state-of-the-art results on how large c can be, the reader is referred to a recent article by Mossinghoff, Trudgian and Yang [44].)

We construct the contour $[\sigma_0 - iT, \sigma_0 + iT, \sigma_1 + iT, \sigma_1 - iT]$, where $\sigma_1 = -\frac{c}{\log^{2/3}(T)(\log\log(T))^{1/3}}$ so that the rectangle formed by the contour completely stays in the Vinogradov-Korobov zero-free region. Invoking the residue theorem and taking into account the simple pole of the integrand at $w = 0$, we see that

$$I = \mathrm{Res}_{w=0}\left(\frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\right) + \int_{\sigma_0+iT}^{\sigma_1+iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\mathrm{d}w + \int_{\sigma_1+iT}^{\sigma_1-iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\mathrm{d}w + \int_{\sigma_1-iT}^{\sigma_0-iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\mathrm{d}w. \tag{4.4}$$

Using the Laurent series expansions of $\zeta(1+2w)$ and $\zeta(1+w)$ around $w = 0$, it is easy to see that

$$\mathrm{Res}_{w=0}\left(\frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\right) = \frac{1}{2}. \tag{4.5}$$

In the given region of integration, we have [61, p. 136]

$$\frac{1}{\zeta(1+\sigma+it)} \ll \log^{\frac{2}{3}}(t)(\log\log(t))^{\frac{1}{3}} \tag{4.6}$$

Also, from [56, p. 98], for $-1/2 \leq \sigma \leq 0, t \geq 2$,

$$|\zeta(1+\sigma+it)| < c_1 t^{100(-\sigma)^{\frac{3}{2}}} \log^{\frac{2}{3}}(t) \tag{4.7}$$

for some $c_1 > 0$. Since $-\sigma < \frac{c}{\log^{2/3}(t)(\log\log(t))^{1/3}}$, we see that $|\zeta(1+\sigma+it)| \ll \log^{\eta}(t)$ for some $\eta > 0$. Thus, combining (4.6) and (4.7), we get

$$\frac{\zeta(1+2w)}{\zeta(1+w)} \ll \log^{\kappa} t \tag{4.8}$$

for some $\kappa > 0$. Hence

$$\int_{\sigma_0+iT}^{\sigma_1+iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\mathrm{d}w \ll \frac{\log^{\kappa} T}{T} x^{\sigma_0}(\sigma_0 - \sigma_1) \ll \frac{\log^{\kappa} T}{T}.$$

Similarly,

$$\int_{\sigma_1-iT}^{\sigma_0-iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\mathrm{d}w \ll \frac{\log^{\kappa} T}{T}.$$

Now we compute

$$\int_{\sigma_1-iT}^{\sigma_1+iT} \frac{\zeta(1+2w)}{\zeta(1+w)} \frac{x^w}{w}\mathrm{d}w.$$

Using (4.8), we obtain

$$\begin{aligned}\int_{\sigma_1-iT}^{\sigma_1+iT}\frac{\zeta(1+2w)}{\zeta(1+w)}\frac{x^w}{w}\mathrm{d}w &\ll x^{\frac{-c_1}{\log^{2/3}(T)(\log\log(T))^{1/3}}}\int_1^T\log^{\kappa}t\frac{\mathrm{d}t}{t}\\ &\ll x^{\frac{-c_1}{\log^{2/3}(T)(\log\log(T))^{1/3}}}\log^{\kappa+1}(T)\end{aligned} \tag{4.9}$$

We now choose $x=\exp(\log^{4/5}T)$. Then from (4.2), (4.4) and (4.5),

$$\sum_{n\le x}\frac{c(n)}{n}=\frac{1}{2}+O\bigg(\frac{2\log^{\kappa}T}{T}+\frac{\log^{\kappa+1}T}{\exp\Big(c_1\log^{2/15}(T)(\log\log(T))^{-1/3}\Big)}+\frac{\exp(0.5\log^{4/5}T)\log T}{T}\bigg).$$

Thus taking the limit $T\to\infty$, we arrive at (4.1). □

Remark 9. *Since $c(n)=\Omega(\sqrt{n})$, neither Ingham's theorem* [19, p. 63] *nor its extended version for bounded arithmetic functions is applicable here. Hence we use the above method of evaluating the infinite series using contour integration. As can be checked, the standard zero-free region does not suffice to have the right-hand sides of* (4.3) *and* (4.9) *go to zero as $T\to\infty$, and hence this is one of those rare instances where one has to resort to the Vinogradov-Korobov zero-free region.*

Theorem 4.2. *With $c(n)$ given by* (1.11)*, we have*

$$\sum_{n=1}^{\infty}\frac{c(n)\log n}{n}=-\frac{\gamma}{2}.$$

Proof. Let $\mathrm{Re}(s)>1$. Differentiating both sides of (1.10) with respect to s, we see that

$$-\sum_{n=1}^{\infty}\frac{c(n)\log n}{n^s}=-\frac{\zeta'(s)\zeta(2s-1)}{\zeta^2(s)}+\frac{2\zeta'(2s-1)}{\zeta(s)}.$$

Again, from [43, p. 140, Corollary 5.3],

$$-\sum_{n\le x}\frac{c(n)\log n}{n}=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\bigg(-\frac{\zeta'(1+w)\zeta(1+2w)}{\zeta^2(1+w)}+\frac{2\zeta'(1+2w)}{\zeta(1+w)}\bigg)\frac{x^w}{w}\mathrm{d}w+R$$

where

$$R\ll\sum_{\substack{x/2<n<2x\\ n\neq x}}\frac{|c(n)|\log n}{n}\min\left(1,\frac{x}{T|x-n|}\right)+\frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|c(n)|\log n}{n^{1+\sigma_0}}.$$

From (1.12), we have $-\sum_{n=1}^{\infty}\frac{|c(n)|\log n}{n^s}=\frac{d}{ds}\left(\frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)}\right)$. Hence $-\sum_{n=1}^{\infty}\frac{|c(n)|\log n}{n^{1+\sigma_0}}\ll\frac{1}{\sigma_0^3}$.

As $c(n)\ll\sqrt{n}$,

$$\sum_{\substack{x/2<n<2x\\ n\neq x}}\frac{|c(n)|\log n}{n}\min\left(1,\frac{x}{T|x-n|}\right)\ll\frac{\sqrt{x}\log^2 x}{T}.$$

So with $\sigma_0=\frac{1}{\log x}$, we obtain

$$R\ll\frac{x^{\sigma_0}}{T\sigma_0^3}+\frac{\sqrt{x}\log^2 x}{T}\ll\frac{\sqrt{x}\log^2 x}{T}.$$

In the Vinogradov-Korobov zero free region, using (4.6), (4.7), and a bound on $\zeta'(1+w)/\zeta(1+w)$ [61, p. 136], we have

$$-\frac{\zeta'(1+w)\zeta(1+2w)}{\zeta^2(1+w)}+\frac{2\zeta'(1+2w)}{\zeta(1+w)}\ll\log^{\kappa}t.$$

for some $\kappa > 0$. Since the rest of the proof follows the same technique as that of Theorem 4.1, we arrive at

$$-\sum_{n=1}^{\infty}\frac{c(n)\log n}{n}=\lim_{w\to 0} w\bigg(-\frac{\zeta'(1+w)\zeta(1+2w)}{\zeta^2(1+w)}+\frac{2\zeta'(1+2w)}{\zeta(1+w)}\bigg)\frac{x^w}{w}=\frac{\gamma}{2}.$$

□

4.2. **The Voronoi summation formula for $\lambda(n)$.** Theorem 2.1 is proved here. Let τ be a number satisfying $0 < \tau < \min(\delta, 1/8)$, where $\delta > 0$ is the number occurring in (2.1).

Define the integral I by

$$I := \frac{1}{2\pi i}\int_{(1+\tau)}\frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s. \tag{4.10}$$

The integral is well-defined since for large enough $|t|$, $\Phi(\sigma+it) \ll t^{-1-\delta}$ and for a fixed $\sigma > 1$, we have $\zeta(2\sigma+2it)/\zeta(\sigma+it) = O_\sigma(1)$. Therefore,

$$I=\frac{1}{2\pi i}\int_{(1+\tau)}\frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s=\sum_{n=1}^{\infty}\frac{\lambda(n)}{2\pi i}\int_{(1+\tau)}\Phi(s)n^{-s}\mathrm{d}s=\sum_{n=1}^{\infty}\lambda(n)\phi(n), \tag{4.11}$$

where the interchange in the order of summation and integration can be justified using the dominated convergence theorem.

We would like to shift the line of integration to $\mathrm{Re}\, s = -\tau$, use the functional equation of $\zeta(s)$, and then employ the change of variable $s \to 1-s$ so as to be able to write the resulting shifted integral as an infinite sum.

Let T be a sufficiently large number. Form the sequence $\{\tilde{T}_n\}_{n=0}^{\infty}$ defined by $\tilde{T}_0 = T$ and $\tilde{T}_n = \tilde{T}_{n-1} + \tilde{T}_{n-1}^{1/3}$ for $n \geq 1$. It is not difficult to see that $\{\tilde{T}_n\}_{n=1}^{\infty} \to \infty$. From [34, Lemma 1] (see also [53, Theorem 2] or [5, Lemma 2.8]), there exists a sequence $\{T_n\}_{n=1}^{\infty}$ with $T_n \in [\tilde{T}_{n-1}, \tilde{T}_n]$ such that

$$\frac{1}{\zeta(\sigma+iT_n)} \ll T_n^{\epsilon} \tag{4.12}$$

for any $\epsilon > 0$ and $1/2 \leq \sigma \leq 2$. From (1.21),

$$\zeta(s)=\chi(s)\zeta(1-s), \tag{4.13}$$

where $\chi(s) := \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)/\Gamma\left(\frac{s}{2}\right)$. Now (3.1) implies

$$\chi(\sigma+it) \sim t^{1/2-\sigma}, \tag{4.14}$$

This, along with (4.13), implies that for $-1 \leq \sigma \leq 1/2$, $1/\zeta(\sigma+iT_n) \ll T_n^{\sigma-\frac{1}{2}+\epsilon} \ll T_n^{\epsilon}$. In conclusion,

$$\frac{1}{\zeta(\sigma+iT_n)} \ll T_n^{\epsilon}, \tag{4.15}$$

for any $\epsilon > 0$ and $-1 \leq \sigma \leq 2$.

Construct the rectangular contour $[1+\tau-iT_n, 1+\tau+iT_n, -\tau+iT_n, -\tau-iT_n]$, where T_n is a number belonging to the sequence $\{T_m\}_{m=1}^{\infty}$ constructed above. Inside the contour lie the pole of the integrand at 0 (due to $\Phi(s)$), the simple pole at 1/2 (due to $\zeta(2s)$) as well as the simple poles due to the non-trivial zeros of $\zeta(s)$ (assuming the simplicity of the zeros). Let ρ_m denote the m^{th} non-trivial zero of $\zeta(s)$. Then, by the residue theorem,

$$\left[\int_{1+\tau-iT_n}^{1+\tau+iT_n}+\int_{1+\tau+iT_n}^{-\tau+iT_n}+\int_{-\tau+iT_n}^{-\tau-iT_n}+\int_{-\tau-iT_n}^{1+\tau-iT_n}\right]\frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s=2\pi i\left(R_0+R_{1/2}+\sum_{|\gamma_m|<T_n}R_{\rho_m}\right), \tag{4.16}$$

where R_a denotes the residue of the integrand $\frac{\zeta(2s)}{\zeta(s)}\Phi(s)$ at the pole a.

Let us analyze the integrals along the horizontal segments as $T_n \to \infty$. For $1/2 \le \sigma < 3$, we have

$$\zeta(\sigma + iT) \ll T^{1/4}, \tag{4.17}$$

as $T \to \infty$, which follows from the bound [61, p. 96, Equation (5.1.8)] $\zeta\left(\frac{1}{2} + iT\right) = O\left(T^{1/4}\right)$ and the Phragmén-Lindelöf principle. Moreover, for $0 \le \sigma < 1/2$, we use (4.13), (4.14), and (4.17) to get $\zeta(\sigma + iT) \ll T^{3/4-\sigma} \ll T^{3/4}$. Now for $-1/4 < \sigma < 0$, we have [61, p. 95, Equation (5.1.3)], $\zeta(\sigma + iT) \ll T^{1/2-\sigma} \ll T^{3/4}$. Combining the last two bounds, for $-1/4 < \sigma < 1/2$, we have

$$\zeta(\sigma + iT) \ll T^{3/4}, \tag{4.18}$$

as $T \to \infty$. Thus, from (4.17) and (4.18),

$$\begin{aligned}\left|\int_{1+\tau+iT_n}^{-\tau+iT_n} \frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s\right| &= \left|\int_{-\tau}^{1+\tau} \frac{\zeta(2u+2iT_n)}{\zeta(u+iT_n)}\Phi(u+iT_n)\mathrm{d}u\right| \\ &\le \left|\int_{-\tau}^{1/4} \frac{\zeta(2u+2iT_n)}{\zeta(u+iT_n)}\Phi(u+iT_n)\mathrm{d}u\right| + \left|\int_{1/4}^{1+\tau} \frac{\zeta(2u+2iT_n)}{\zeta(u+iT_n)}\Phi(u+iT_n)\mathrm{d}u\right| \\ &\ll \int_{-\tau}^{1/4} T_n^{3/4}T_n^{\epsilon}T_n^{-1-\delta}\mathrm{d}u + \int_{1/4}^{1+\tau} T_n^{1/4}T_n^{\epsilon}T_n^{-1-\delta}\mathrm{d}u.\end{aligned}$$

Now choose $\epsilon = \delta$ so that

$$\lim_{T_n\to\infty} \int_{1/4}^{1+\tau} T_n^{1/4}T_n^{\epsilon}T_n^{-1-\delta}\mathrm{d}u = \lim_{T_n\to\infty} \left(\frac{3}{4} + \tau\right) T_n^{-3/4} = 0,$$

and $\int_{-\tau}^{1/4} T_n^{3/4}T_n^{\epsilon}T_n^{-1-\delta}\mathrm{d}u \ll T_n^{-1/4}$ so that

$$\lim_{T_n\to\infty} \int_{-\tau}^{1/4} T_n^{1/2-2u}T_n^{\epsilon}T_n^{-1-\delta}\mathrm{d}u = 0.$$

Therefore,

$$\lim_{T_n\to\infty} \int_{1+\tau+iT_n}^{-\tau+iT_n} \frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s = 0.$$

Similarly, the integral along $[-\tau - iT_n, 1 + \tau - iT_n]$ can be shown to approach zero as $T_n \to \infty$. Moreover, observe that using the functional equation (4.13), we have, on the line $\mathrm{Re}(s) = -\tau$, $\zeta(2s)/\zeta(s) \ll_\tau t^\tau$, which, along with the hypothesis $\Phi(s) \ll t^{-1-\delta}$ and the fact that $\tau < \delta$, implies

$$\int_{(-\tau)} \frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s \ll 1. \tag{4.19}$$

Thus, from (4.10), (4.16), (4.19) and the functional equation (4.13), we see that

$$\begin{aligned}I &= \frac{1}{2\pi i}\int_{(-\tau)} \frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s + R_0 + R_{1/2} + \lim_{T_n\to\infty} \sum_{|\gamma_m|<T_n} R_{\rho_m} &(4.20)\\ &= \frac{1}{2\pi i}\int_{(-\tau)} \frac{\zeta(1-2s)\Gamma(\frac{1}{2}-s)\Gamma(\frac{s}{2})}{\zeta(1-s)\Gamma(s)\Gamma(\frac{1-s}{2})}\Phi(s)\pi^s\mathrm{d}s + R_0 + R_{1/2} + \lim_{T_n\to\infty} \sum_{|\gamma_m|<T_n} R_{\rho_m} \\ &= \frac{1}{2\pi i}\int_{(1+\tau)} \frac{\zeta(2s-1)\Gamma(s-\frac{1}{2})\Gamma(\frac{1-s}{2})}{\zeta(s)\Gamma(1-s)\Gamma(\frac{s}{2})}\Phi(1-s)\pi^{1-s}\mathrm{d}s + R_0 + R_{1/2} + \lim_{T_n\to\infty} \sum_{|\gamma_m|<T_n} R_{\rho_m}, &(4.21)\end{aligned}$$

where we employed the change of variable $s \to 1 - s$. It is easy to see that

$$R_{\frac{1}{2}} = \lim_{s\to\frac{1}{2}} \left(s - \frac{1}{2}\right)\frac{\zeta(2s)}{\zeta(s)}\Phi(s) = \lim_{s\to\frac{1}{2}} \left(s - \frac{1}{2}\right)\frac{\zeta(2s)}{\zeta(s)}\int_0^\infty \phi(x)x^{s-1}\mathrm{d}x = \frac{1}{2\zeta(\frac{1}{2})}\int_0^\infty \frac{\phi(x)}{\sqrt{x}}\mathrm{d}x,$$

$$R_{\rho_m} = \lim_{s\to\rho_m}(s-\rho_m)\frac{\zeta(2s)}{\zeta(s)}\Phi(s) = \lim_{s\to\rho_m}(s-\rho_m)\frac{\zeta(2s)}{\zeta(s)}\int_0^\infty \phi(x)x^{s-1}\mathrm{d}x = \frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\int_0^\infty \phi(x)x^{\rho_m-1}\mathrm{d}x,$$
$$R_0 = \mathrm{Res}_{s=0}\frac{\zeta(2s)}{\zeta(s)}\Phi(s), \tag{4.22}$$

where, in the first two residue calculations, we invoked the Mellin inversion theorem [41, p. 341, Theorem 1]. We choose to simplify the expression for R_0 at the end.

It remains to analyze the integral on the right-hand side of (4.21), which we denote by I_1. Observe that using (1.10), we have

$$\begin{aligned}I_1 &= \frac{\pi}{2\pi i}\sum_{n=1}^\infty c(n)\int_{(1+\tau)}\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)\Gamma(\frac{s}{2})}\Phi(1-s)(\pi n)^{-s}\mathrm{d}s\\ &= \frac{\sqrt{\pi}}{2\pi i}\sum_{n=1}^\infty c(n)\int_{(1+\tau)}\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\Phi(1-s)2^s(\pi n)^{-s}\mathrm{d}s\\ &= \sqrt{\pi}\sum_{n=1}^\infty c(n)J\left(\frac{\pi n}{2}\right),\end{aligned} \tag{4.23}$$

where

$$J(x) := \frac{1}{2\pi i}\int_{(1+\tau)}\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\Phi(1-s)x^{-s}\mathrm{d}s.$$

The interchange of the order of summation and integration can be justified using the dominated convergence theorem. We would like to express $J(x)$ as a real integral using Parseval's formula (3.2). However, from the conditions on Φ, it can be seen that $\Phi(1-s)$ is analytic in $-1<\mathrm{Re}(s)<1$ whereas $\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)x^{-s}$ is analytic in $-1/2<\mathrm{Re}(s)<1/2$. Hence the Parseval formula is inapplicable as $\mathrm{Re}(s)>1$ in the definition of $J(x)$. To address this issue, we shift the line of integration to from $\mathrm{Re}(s)=1+\tau$ to $\mathrm{Re}(s)=-\tau$, and then employ change of variable s to $1-s$ as will be seen below.

Using the reflection formula for Gamma function and (3.1), it is easy to see that as $t\to\pm\infty$,

$$\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right) \ll t^{\sigma-1}. \tag{4.24}$$

Also, $\Phi(1-s)\ll t^{-1-\delta}$. This implies that the integrals along the horizontal segments tend to zero as $t\to\pm\infty$. Therefore, by Cauchy's residue theorem,

$$\begin{aligned}J(x) &= \frac{1}{2\pi i}\int_{(-\tau)}\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\Phi(1-s)x^{-s}\mathrm{d}s + \tilde{R}_{\frac{1}{2}}(x)+\tilde{R}_1(x)\\ &= \frac{1}{2\pi i}\int_{(1+\tau)}\Gamma\left(\frac{1}{2}-s\right)\cos\left(\frac{\pi s}{2}\right)\Phi(s)x^{s-1}\mathrm{d}s + \tilde{R}_{\frac{1}{2}}(x)+\tilde{R}_1(x),\end{aligned} \tag{4.25}$$

where $\tilde{R}_a(x)$ denotes the residue of the integrand $\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\Phi(1-s)x^{-s}$ at the pole a, and where, in the last step, we made the change of variable $s\to 1-s$.

We now justify the applicability of Parseval's formula to the integral on the last line of (4.25). Let $F(s)=\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)x^{-s}$ and $G(s)=\Phi(s)$. Firstly, the line of integration of the integral on the right-hand side of (4.25) lies in the common strip of analyticity of $F(1-s)$ and $G(s)$, that is, $1/2<\mathrm{Re}(s)<3/2$. Moreover, it is clear from (4.24) that $F(-\tau-it)\in L(-\infty,\infty)$, and also, the fact that $\Phi(s)$ is holomorphic in $0<\mathrm{Re}(s)<2$ implies $t^\tau\phi(t)\in L(0,\infty)$. Hence by Parseval's formula 3.2,

$$\frac{1}{2\pi i}\int_{(1+\tau)}\Gamma\left(\frac{1}{2}-s\right)\cos\left(\frac{\pi s}{2}\right)\Phi(s)x^{s-1}\mathrm{d}s = \int_0^\infty \phi(t)\left(\sin\left(tx+\frac{\pi}{4}\right)(tx)^{-1/2}-\frac{(tx)^{-1/2}}{\sqrt{2}}\right)\mathrm{d}t, \tag{4.26}$$

since

$$\frac{1}{2\pi i}\int_{(-\tau)}F(s)t^{-s}\mathrm{d}s=\sin\left(tx+\frac{\pi}{4}\right)(xt)^{-1/2}-\frac{(xt)^{-1/2}}{\sqrt{2}},\tag{4.27}$$

which is what we set to prove next. Employing the change of variable $s=w+1/2$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(-\tau)}\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)t^{-s}\mathrm{d}s\\&=\frac{t^{-1/2}}{\sqrt{2}}\left\{\frac{1}{2\pi i}\int_{(-\tau-\frac{1}{2})}\Gamma(w)\sin\left(\frac{\pi w}{2}\right)t^{-w}\mathrm{d}w+\frac{1}{2\pi i}\int_{(-\tau-\frac{1}{2})}\Gamma(w)\cos\left(\frac{\pi w}{2}\right)t^{-w}\mathrm{d}w\right\}.\end{aligned}\tag{4.28}$$

Now we know that for $-1/2<\mathrm{Re}(\xi)<1/2$,

$$\frac{1}{2\pi i}\int_{(\mathrm{Re}(\xi))}\Gamma(w)\sin\left(\frac{\pi w}{2}\right)t^{-w}\mathrm{d}w=\sin(t),\tag{4.29}$$

and for $0<\mathrm{Re}(\xi)<1/2$,

$$\frac{1}{2\pi i}\int_{(\mathrm{Re}(\xi))}\Gamma(w)\cos\left(\frac{\pi w}{2}\right)t^{-w}\mathrm{d}w=\cos(t).\tag{4.30}$$

Thus, in order to use both (4.29) and (4.30) in (4.28), we need to shift the line of integration from $\mathrm{Re}(w)=-\tau-1/2$ to $0<\mathrm{Re}(w)=\lambda<1/2$ in each of the integrals on the right-hand side of (4.28). While this produces no poles for the first integral, we do have to consider the contribution of pole at $w=0$ for the second integral at which the residue is 1. Since the integrals along the horizontal segments in each of the integrals go to zero as $T\to\infty$, combining all of this and finally replacing t by xt, we obtain (4.27). Further,

$$\tilde{R}_{\frac{1}{2}}(x)=\frac{x^{-1/2}}{\sqrt{2}}\Phi\left(\frac{1}{2}\right).\tag{4.31}$$

Hence from (4.25), (4.26) and (4.31), we have

$$\begin{aligned}J(x)&=\int_0^\infty\phi(t)\left(\sin\left(tx+\frac{\pi}{4}\right)(tx)^{-1/2}-\frac{(tx)^{-1/2}}{\sqrt{2}}\right)\mathrm{d}t+\frac{x^{-1/2}}{\sqrt{2}}\Phi\left(\frac{1}{2}\right)+\tilde{R}_1(x)\\&=x^{-1/2}\int_0^\infty\frac{\phi(t)}{\sqrt{t}}\sin\left(tx+\frac{\pi}{4}\right)\mathrm{d}t+\tilde{R}_1(x),\end{aligned}$$

since $\Phi(1/2)=\int_0^\infty\phi(t)/\sqrt{t}\,\mathrm{d}t$. Along with the definition of $\tilde{R}_1(x)$, this gives

$$\begin{aligned}J\left(\frac{\pi n}{2}\right)&=\sqrt{\frac{2}{\pi n}}\int_0^\infty\frac{\phi(t)}{\sqrt{t}}\sin\left(\frac{\pi nt}{2}+\frac{\pi}{4}\right)\mathrm{d}t+\mathrm{Res}_{s=1}\,\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\Phi(1-s)\left(\frac{\pi n}{2}\right)^{-s}\\&=\sqrt{\frac{2}{\pi n}}\int_0^\infty\frac{\phi(t)}{\sqrt{t}}\sin\left(\frac{\pi nt}{2}+\frac{\pi}{4}\right)\mathrm{d}t-\frac{2}{\pi n}\,\mathrm{Res}_{s=0}\,\Gamma\left(\frac{1}{2}-s\right)\cos\left(\frac{\pi s}{2}\right)\Phi(s)\left(\frac{\pi n}{2}\right)^{s}.\end{aligned}\tag{4.32}$$

We now evaluate the residue on the extreme right-hand side of the above equation. As $s\to0$, the standard power series expansions of the Gamma function, cosine and the power function give

$$\Gamma\left(\frac{1}{2}-s\right)\cos\left(\frac{\pi s}{2}\right)\left(\frac{\pi n}{2}\right)^{s}=\sqrt{\pi}\left\{1+\left(\log n+\log\left(\frac{\pi}{2}\right)-\psi\left(\frac{1}{2}\right)\right)s+O(|s|^2)\right\},$$

where $\psi(s)$ is the digamma function. Since we have assumed $\Phi(s)$ has at most a double pole at $s=0$, let

$$\Phi(s)=\sum_{m=-2}^{\infty}a_ms^m.$$

Then it can be seen that

$$\text{Res}_{s=0}\Gamma\left(\frac{1}{2}-s\right)\cos\left(\frac{\pi s}{2}\right)\Phi(s)\left(\frac{\pi n}{2}\right)^{s}=\sqrt{\pi}\left(a_{-2}\left(\log n+\log\left(\frac{\pi}{2}\right)-\psi\left(\frac{1}{2}\right)\right)+a_{-1}\right). \tag{4.33}$$

Therefore, from (4.23), (4.32) and (4.33),

$$\begin{aligned}I_1&=\sqrt{2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\int_0^{\infty}\frac{\phi(t)}{\sqrt{t}}\sin\left(\frac{\pi nt}{2}+\frac{\pi}{4}\right)\mathrm{d}t-2\sum_{n=1}^{\infty}\frac{c(n)}{n}\left(a_{-2}\left(\log n+\log\left(\frac{\pi}{2}\right)-\psi\left(\frac{1}{2}\right)\right)+a_{-1}\right)\\&=\sqrt{2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\int_0^{\infty}\frac{\phi(t)}{\sqrt{t}}\sin\left(\frac{\pi nt}{2}+\frac{\pi}{4}\right)\mathrm{d}t-\log(2\pi)\cdot a_{-2}-a_{-1},\end{aligned} \tag{4.34}$$

where, in the last step, we used Theorems 4.1, 4.2 and the fact [26, p. 905, formula **8.366.2**] that $\psi(1/2)=-\gamma-2\log(2)$. Thus from (4.21)-(4.23) and (4.34),

$$\begin{aligned}I=&\sqrt{2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\int_0^{\infty}\frac{\phi(t)}{\sqrt{t}}\sin\left(\frac{\pi nt}{2}+\frac{\pi}{4}\right)\mathrm{d}t-\log(2\pi)\cdot a_{-2}-a_{-1}+\frac{1}{2\zeta(\frac{1}{2})}\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\mathrm{d}x\\&+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\int_0^{\infty}\phi(x)x^{\rho_m-1}\mathrm{d}x+R_0.\end{aligned} \tag{4.35}$$

The only thing remaining now is to evaluate the expression for R_0 given in (4.22). To that end, as $s\to 0$, we have from [61, pp. 19-20, Equations (2.4.3), (2.4.5)], $\zeta(s)=-\frac{1}{2}-\frac{1}{2}\log(2\pi)s+O(|s|^2)$, which implies $\frac{\zeta(2s)}{\zeta(s)}=1+\log(2\pi)s+O(|s|^2)$ so that R_0 is the coefficient of s^{-1} in

$$\left(1+\log(2\pi)s+O(|s|^2)\right)\left(\frac{a_{-2}}{s^2}+\frac{a_{-1}}{s}+O(1)\right).$$

This implies $R_0=\log(2\pi)\cdot a_{-2}+a_{-1}$. Substituting this expression in (4.35) , simplifying, and equating the resulting expression for I with the right-hand side of (4.11), we arrive at (2.2). This completes the proof.

4.3. **Corollaries of the Voronoi summation formula.**

Proof of Corollary 2.2. Let $\mathrm{Re}(s)>0$ and $y>0$. Let $\Phi(s)=\Gamma(s)y^{-s}$ in Theorem 2.1. Firstly, $\Phi(s)$ satisfies the conditions given before the statement of Theorem 2.1 and has a simple pole at $s=0$. Secondly, $\phi(x)=e^{-xy}$. Also, the following is easily verified:

$$\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\mathrm{d}x=\frac{\sqrt{\pi}}{\sqrt{y}},\qquad\int_0^{\infty}\phi(x)x^{\rho_m-1}\mathrm{d}x=\frac{\Gamma(\rho_m)}{y^{\rho_m}}.$$

Next,

$$\begin{aligned}\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}+\frac{\pi}{4}\right)\mathrm{d}x&=\frac{1}{\sqrt{2}}\left(\int_0^{\infty}\frac{e^{-xy}}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}\right)\mathrm{d}x+\int_0^{\infty}\frac{e^{-xy}}{\sqrt{x}}\cos\left(\frac{\pi nx}{2}\right)\mathrm{d}x\right)\\&=\frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\pi}{2}}\frac{\sqrt{\sqrt{y^2+\frac{\pi^2n^2}{4}}-y}}{\sqrt{y^2+\frac{\pi^2n^2}{4}}}+\sqrt{\frac{\pi}{2}}\frac{\sqrt{\sqrt{y^2+\frac{\pi^2n^2}{4}}+y}}{\sqrt{y^2+\frac{\pi^2n^2}{4}}}\right\}\\&=\frac{\sqrt{\pi}}{2}\frac{\left(\sqrt{\sqrt{y^2+\frac{\pi^2n^2}{4}}-y}+\sqrt{\sqrt{y^2+\frac{\pi^2n^2}{4}}+y}\right)}{\sqrt{y^2+\frac{\pi^2n^2}{4}}},\end{aligned}$$

where, in the second step, we used the integral evaluations [26, p. 499, Equations **3.944.13, 3.944.14**] with $\mu=1/2,n=0,\beta=y$ and $b=\pi n/2$. Using all of these evaluations, we arrive at (2.3).

To prove (2.4), we first observe that as $n\to\infty$,

$$\frac{\sqrt{\sqrt{4y^2+\pi^2n^2}-2y}+\sqrt{\sqrt{4y^2+\pi^2n^2}+2y}}{\sqrt{4y^2+\pi^2n^2}}=\frac{2}{\sqrt{\pi n}}+O_y\left(\frac{1}{n^{5/2}}\right),$$

where the big-O involves terms with positive powers of y. Therefore, using (4.1), we see that as $y\to 0^+$,

$$\sqrt{\pi}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\frac{\sqrt{\sqrt{4y^2+\pi^2n^2}-2y}+\sqrt{\sqrt{4y^2+\pi^2n^2}+2y}}{\sqrt{4y^2+\pi^2n^2}}=O(1).$$

Since we assume the RH and the absolute convergence of $\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)\Gamma(\rho_m)}{\zeta'(\rho_m)}y^{-\rho_m}$, we see that

$$\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)\Gamma(\rho_m)}{\zeta'(\rho_m)}y^{-\rho_m}\ll y^{-1/2}\sum_{\rho_m}\left|\frac{\zeta(2\rho_m)\Gamma(\rho_m)}{\zeta'(\rho_m)}\right|=O(y^{-1/2}).$$

Along with (2.3), the above two equations imply (2.4). □

Proof of Corollary 2.3. Let $\mathrm{Re}(s)>0$ and $y>0$. Let $\Phi(s)=\frac{1}{2}\Gamma\left(\frac{s}{2}\right)y^{-\frac{s}{2}}$ in Theorem 2.1 so that $\phi(x)=e^{-x^2y}$. We refrain from giving the details since they are similar to that of Corollary 2.2; instead, we only show below how one obtains the evaluation

$$\int_0^\infty\frac{e^{-x^2y}}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}+\frac{\pi}{4}\right)\mathrm{d}x=\frac{\pi^{3/2}\sqrt{n}}{4\sqrt{2y}}e^{-\frac{\pi^2n^2}{32y}}\left(I_{-\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)+I_{\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)\right).\tag{4.36}$$

From [26, p. 503, formulas **3.952.7, 3.952.8**],

$$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\sin(cx)\mathrm{d}x=\frac{ce^{-\frac{c^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}}\Gamma\left(\frac{1+\mu}{2}\right){}_1F_1\left(1-\frac{\mu}{2};\frac{3}{2};\frac{c^2}{4\beta}\right)\qquad(\mathrm{Re}(\beta)>0,\mathrm{Re}(\mu)>-1),\tag{4.37}$$

$$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\cos(cx)\mathrm{d}x=\frac{e^{-\frac{c^2}{4\beta}}}{2\beta^{\frac{\mu}{2}}}\Gamma\left(\frac{\mu}{2}\right){}_1F_1\left(\frac{1-\mu}{2};\frac{1}{2};\frac{c^2}{4\beta}\right)\qquad(\mathrm{Re}(\beta)>0,\mathrm{Re}(\mu)>0).\tag{4.38}$$

Let $\mu=1/2,\beta=y$ and $c=\pi n/2$ in (4.37) to get

$$\int_0^\infty\frac{e^{-x^2y}}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}\right)dx=\frac{\pi n}{4y^{3/4}}\Gamma\left(\frac{3}{4}\right)e^{-\frac{\pi^2n^2}{16y}}{}_1F_1\left(\frac{3}{4};\frac{3}{2};\frac{\pi^2n^2}{16y}\right).\tag{4.39}$$

Now from [52, p. 126, Theorem 43], for $2a$ not equal to a negative odd integer,

$$e^{-z}{}_1F_1(a;2a;2z)={}_0F_1\left(-;a+\tfrac{1}{2};\tfrac{z^2}{4}\right)=\Gamma\left(a+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{\frac{1}{2}-a}I_{a-\frac{1}{2}}(z),$$

where the second equality follows from (1.2) and (1.4). Using the above equation with $a=3/4$ and $z=\pi^2n^2/(32y)$ and using the resultant in (4.39), we arrive at

$$\int_0^\infty\frac{e^{-x^2y}}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}\right)dx=\frac{\pi^{3/2}\sqrt{n}}{4\sqrt{y}}e^{-\frac{\pi^2n^2}{32y}}I_{\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right).\tag{4.40}$$

Similarly letting $\mu=1/2,\beta=y$ and $c=\pi n/2$ in (4.38), we get

$$\int_0^\infty\frac{e^{-x^2y}}{\sqrt{x}}\cos\left(\frac{\pi nx}{2}\right)dx=\frac{\pi^{3/2}\sqrt{n}}{4\sqrt{y}}e^{-\frac{\pi^2n^2}{32y}}I_{-\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right).\tag{4.41}$$

Therefore, (4.40) and (4.41) result in (4.36), which, in turn, leads to (2.5).

Using the asymptotic expansion of the modified Bessel function of the second kind [64, p. 203, Equation (7.23.3)], it can be checked that as $n\to\infty$,

$$e^{-\frac{\pi^2n^2}{32y}}\left(I_{-\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)+I_{\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)\right)=\frac{8\sqrt{y}}{\pi^{3/2}n}\left(1+O_y\left(\frac{1}{n^2}\right)\right),$$

where the big-O term involves positive powers of y. Along with (4.1), this implies that

$$\frac{\pi^{3/2}}{4\sqrt{y}}\sum_{n=1}^{\infty}c(n)e^{-\frac{\pi^2n^2}{32y}}\left(I_{-\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)+I_{\frac{1}{4}}\left(\frac{\pi^2n^2}{32y}\right)\right)=O(1),$$

which leads to (2.6) following the analysis similar to that done for obtaining (2.4). □

Proof of Corollary 2.4. Let $\mathrm{Re}(s)>0$ and $y>0$, and let $\Phi(s):=2^{s-2}y^{-s}\Gamma^2\left(\frac{s}{2}\right)$ in Theorem 2.1. Then $\Phi(s)$ has a second order pole at $s=0$, and it satisfies the conditions given before the statement of Theorem 2.1. Also, from [46, p. 196, formula 5.39], we have $\phi(x)=K_0(xy)$. Next,

$$\int_0^\infty\frac{\phi(x)}{\sqrt{x}}\mathrm{d}x=\frac{4\sqrt{2}}{\sqrt{y}}\Gamma^2\left(\frac{5}{4}\right),\qquad\int_0^\infty\phi(x)x^{\rho_m-1}\mathrm{d}x=2^{\rho_m-2}y^{-\rho_m}\Gamma^2\left(\frac{\rho_m}{2}\right).$$

Thus we will be done with proving (2.7) provided we show that

$$\int_0^\infty\frac{\phi(x)}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}+\frac{\pi}{4}\right)\mathrm{d}x=\frac{\pi^{\frac{3}{2}}}{\sqrt{2}(\pi^2n^2+4y^2)^{\frac{1}{4}}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{\pi n}{2\sqrt{\pi^2n^2+4y^2}}\right).\tag{4.42}$$

To that end, using [26, p. 731, formulas **6.699.3, 6.699.4**], we have

$$\int_0^\infty\frac{\phi(x)}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}\right)\mathrm{d}x=\frac{\pi n}{(2y)^{3/2}}\Gamma^2\left(\frac{3}{4}\right){}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};-\frac{\pi^2n^2}{4y^2}\right),\tag{4.43}$$

$$\int_0^\infty\frac{\phi(x)}{\sqrt{x}}\cos\left(\frac{\pi nx}{2}\right)\mathrm{d}x=\frac{1}{2^{3/2}\sqrt{y}}\Gamma^2\left(\frac{1}{4}\right){}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};-\frac{\pi^2n^2}{4y^2}\right),\tag{4.44}$$

Using [2, p. 176, Exercise 1(e)], with $a=b=1/4$, we obtain

$${}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};-x\right)=\frac{(1+x)^{-1/4}}{2\sqrt{\pi}}\Gamma^2\left(\frac{3}{4}\right)\left\{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{1}{2}\sqrt{\frac{x}{1+x}}\right)+{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}-\frac{1}{2}\sqrt{\frac{x}{1+x}}\right)\right\}.\tag{4.45}$$

Similarly, one can derive (see, for example, [65][5]

$${}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};-x\right)=\frac{(1+x)^{-1/4}}{4\sqrt{\pi x}}\Gamma^2\left(\frac{1}{4}\right)\left\{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{1}{2}\sqrt{\frac{x}{1+x}}\right)-{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}-\frac{1}{2}\sqrt{\frac{x}{1+x}}\right)\right\}.\tag{4.46}$$

From (4.45) and (4.46), it is easy to derive

$$2\sqrt{x}\Gamma^2\left(\frac{3}{4}\right){}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};-x\right)+\Gamma^2\left(\frac{1}{4}\right){}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};-x\right)=\frac{2\pi^{3/2}}{(1+x)^{1/4}}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{1}{2}\sqrt{\frac{x}{1+x}}\right).\tag{4.47}$$

Now let $x=\pi^2n^2/(4y^2)$ in (4.47), divide both sides by $2^{3/2}\sqrt{y}$, and then compare the left-hand side of the resulting equation with the right-hand side of the equation obtained by adding the corresponding sides of (4.43) and (4.44). This establishes (4.42) which completes the proof of (2.7).

To prove (2.8), observe that as $n\to\infty$,

$$\frac{1}{(\pi^2n^2+4y^2)^{\frac{1}{4}}}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{\pi n}{2\sqrt{\pi^2n^2+4y^2}}\right)=\frac{2\log\left(4\pi n/y\right)}{\pi^{3/2}\sqrt{n}}+O_y\left(n^{-5/2}\right),$$

[5]It is to be noted that the notation for the complete elliptic integral used here is such that $K(k)=\frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k\right)$, which is different from the conventional notation whereby we have $K(k)=\frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$.

where the big-O term involves positive powers of y. Thus, using Theorems 4.1 and 4.2, we see that as $y\to 0$,

$$\pi^{\frac{3}{2}}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}(\pi^2n^2+4y^2)^{\frac{1}{4}}}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{\pi n}{2\sqrt{\pi^2n^2+4y^2}}\right)=O\left(\frac{1}{\sqrt{y}}\right).$$

Now RH and the assumption on the absolute convergence of the series in (2.7) involving the non-trivial zeros of $\zeta(s)$ imply (2.8). □

Proof of Corollary 2.5. Let $\mathrm{Re}(s)>0$, $y>0$, and let $\Phi(s):=\frac{\sqrt{\pi}y^{s+\frac{1}{2}}\Gamma(s)}{2\Gamma\left(s+\frac{3}{2}\right)}$. Then $\Phi(s)$ has a simple pole at $s=0$. Also from [46, p. 195, formula 5.35], $\phi(x)=(y-x)^{1/2}$ if $x<y$, and 0 else. Then

$$\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\mathrm{d}x=\frac{\pi y}{2},\qquad \int_0^{\infty}\phi(x)x^{\rho_m-1}\mathrm{d}x=\frac{\sqrt{\pi}y^{\rho_m+\frac{1}{2}}\Gamma(\rho_m)}{2\Gamma\left(\rho_m+\frac{3}{2}\right)}. \tag{4.48}$$

We next show

$$\begin{aligned}\int_0^{\infty}\frac{\phi(x)}{\sqrt{x}}\sin\left(\frac{\pi nx}{2}+\frac{\pi}{4}\right)\mathrm{d}x=\frac{\pi y}{2\sqrt{2}}\bigg\{&J_0\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)+\cos\left(\frac{\pi ny}{4}\right)\right)\\&+J_1\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)-\cos\left(\frac{\pi ny}{4}\right)\right)\bigg\}.\end{aligned} \tag{4.49}$$

To that end, letting $\alpha=1/2,\beta=3/2,b=\pi n/2,\delta=1$ in [49, p. 391, Formula **2.5.7.1**], one finds that

$$\int_0^{y}x^{-1/2}(y-x)^{1/2}\sin\left(\frac{\pi nx}{2}\right)\,dx=-\frac{i\pi y}{4}\left\{{}_1F_1\left(\frac{1}{2};2;\frac{i\pi ny}{2}\right)-{}_1F_1\left(\frac{1}{2};2;-\frac{i\pi ny}{2}\right)\right\}. \tag{4.50}$$

Now from [51, p. 580, Formula **7.11.2.12**],

$${}_1F_1\left(\frac{1}{2};2;z\right)=e^{z/2}\left(I_0\left(\frac{z}{2}\right)-I_1\left(\frac{z}{2}\right)\right),$$

so that

$${}_1F_1\left(\frac{1}{2};2;iz\right)=e^{iz/2}\left(J_0\left(\frac{z}{2}\right)-iJ_1\left(\frac{z}{2}\right)\right),$$

and thus

$${}_1F_1\left(\frac{1}{2};2;iz\right)-{}_1F_1\left(\frac{1}{2};2;-iz\right)=2i\left(J_0\left(\frac{z}{2}\right)\sin\left(\frac{z}{2}\right)-J_1\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)\right). \tag{4.51}$$

Therefore, from (4.50) and (4.51),

$$\int_0^{y}x^{-1/2}(y-x)^{1/2}\sin\left(\frac{\pi nx}{2}\right)\,dx=\frac{\pi y}{2}\left(J_0\left(\frac{\pi ny}{4}\right)\sin\left(\frac{\pi ny}{4}\right)-J_1\left(\frac{\pi ny}{4}\right)\cos\left(\frac{\pi ny}{4}\right)\right).$$

Similarly using [49, p. 391, Formula **2.5.7.1**] with $\alpha=1/2,\beta=3/2,b=\pi n/2,\delta=0$, it can be seen that

$$\int_0^{y}x^{-1/2}(y-x)^{1/2}\cos\left(\frac{\pi nx}{2}\right)\,dx=\frac{\pi y}{2}\left(J_0\left(\frac{\pi ny}{4}\right)\cos\left(\frac{\pi ny}{4}\right)+J_1\left(\frac{\pi ny}{4}\right)\sin\left(\frac{\pi ny}{4}\right)\right).$$

The above two equations establish (4.49). Now use (4.48) and (4.49) in Theorem 2.1 and divide both sides of the resulting equation by $\sqrt{y}$ to arrive at (2.9).

To prove (2.10), we first note the well-known asymptotic expansion of $J_\nu(z)$ [64, p. 199] as $|z|\to\infty$:

$$J_\nu(z)\sim\sqrt{\frac{2}{\pi z}}\left(\cos(w)\sum_{n=0}^{\infty}\frac{(-1)^n(\nu,2n)}{(2z)^{2n}}-\sin(w)\sum_{n=0}^{\infty}\frac{(-1)^n(\nu,2n+1)}{(2z)^{2n+1}}\right),\quad (|\arg(z)|<\pi),$$

where $w = z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi$ and $(\nu, n) = \frac{\Gamma(\nu+n+1/2)}{\Gamma(n+1)\Gamma(\nu-n+1/2)}$. Then a simple calculation leads to the fact that as $n \to \infty$,

$$J_0\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)+\cos\left(\frac{\pi ny}{4}\right)\right)+J_1\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)-\cos\left(\frac{\pi ny}{4}\right)\right)=\frac{4}{\pi\sqrt{ny}}\left(1+O_y\left(n^{-\frac{3}{2}}\right)\right),$$

where the big-O term involves terms with negative powers of y. Thus, using Theorem 4.1, as $y \to \infty$,

$$\frac{\pi\sqrt{y}}{2}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}\left\{J_0\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)+\cos\left(\frac{\pi ny}{4}\right)\right)+J_1\left(\frac{\pi ny}{4}\right)\left(\sin\left(\frac{\pi ny}{4}\right)-\cos\left(\frac{\pi ny}{4}\right)\right)\right\}$$
$$=1+O\left(\frac{1}{y}\right).$$

Now RH and the assumption on the absolute convergence of the series in (2.9) involving the non-trivial zeros of $\zeta(s)$ imply (2.10). □

4.4. **Cohen type identity for $\lambda(n)$.** Theorem 2.6 is proved here. Let $c = \mathrm{Re}(s) > 1$. We do this by evaluating the integral

$$I := \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(2s-1)\zeta(2s-1)}{\Gamma(s)\zeta(s)}(2\pi x)^{-s}\mathrm{d}s \tag{4.52}$$

in two different ways. On one hand, using (1.10) in the first step and duplication formula in the second, we have

$$\begin{aligned}I &= \sum_{n=1}^{\infty}c(n)\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(2s-1)}{\Gamma(s)}(2\pi nx)^{-s}\mathrm{d}s\\ &= \frac{1}{4\sqrt{\pi}}\sum_{n=1}^{\infty}c(n)\frac{1}{2\pi i}\int_{(c)}\Gamma\left(s-\frac{1}{2}\right)\left(\frac{\pi nx}{2}\right)^{-s}\mathrm{d}s\\ &= \frac{1}{2\pi\sqrt{2x}}\sum_{n=1}^{\infty}\frac{c(n)}{\sqrt{n}}e^{-\pi nx/2}.\end{aligned} \tag{4.53}$$

On the other hand, using (1.20) twice, we see that

$$I = \frac{1}{4\pi^2 i}\int_{(c)}\frac{\zeta(2-2s)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)}\mathrm{d}s.$$

Next, in order to use the fact that for $\mathrm{Re}(s) < 0$, $\sum_{n=1}^{\infty}\lambda(n)n^{s-1} = \zeta(2-2s)/\zeta(1-s)$, we shift the line of integration from $c = \mathrm{Re}(s) > 1$ to $-2 < c' = \mathrm{Re}(s) < 0$ by constructing a rectangular contour $[c - iT_n, c + iT_n, c' + iT_n, c' - iT_n, c - iT_n]$, where $\{T_n\}_{n=1}^{\infty}$ is the sequence constructed in the proof of Theorem 2.1 which gives (4.12). Along with (3.1), this shows that the integrals along the horizontal segments tend to zero as $n \to \infty$. We need to consider the contribution of the poles of the integrand in (4.52)at 1/2 (due to $\Gamma(2s-1)$) and at the non-trivial zeros of $\zeta(s)$ at $s = \rho_m, m \in \mathbb{Z}$. The residues at these poles are given by

$$R_{1/2} = \frac{-1}{4\pi\sqrt{2x}\zeta(1/2)}, \qquad R_{\rho_m} = \frac{\zeta(2\rho_m-1)\Gamma(2\rho_m-1)}{\zeta'(\rho_m)\Gamma(\rho_m)}(2\pi x)^{-\rho_m}. \tag{4.54}$$

Therefore, by the residue theorem,

$$\begin{aligned}I &= \frac{1}{4\pi^2 i}\int_{(c')}\frac{x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{1-s}}\mathrm{d}s + R_{1/2} + \lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}R_{\rho_m}\\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}\frac{1}{2\pi i}\int_{(c')}\frac{(x/n)^{-s}\mathrm{d}s}{2\sin\left(\frac{\pi s}{2}\right)} + R_{1/2} + \lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}R_{\rho_m}.\end{aligned} \tag{4.55}$$

In order to use the well-known formula,

$$\frac{1}{2\pi i}\int_{(\mathrm{Re}(s))}\frac{x^{-s}\mathrm{d}s}{2\sin\left(\frac{\pi s}{2}\right)}=\frac{1}{\pi(1+x^2)},$$

which is valid for $0<\mathrm{Re}(s)<2$, we again shift the line of integration from $\mathrm{Re}(s)=c'$ to $0<\mathrm{Re}(s)<2$, consider the contribution of the pole at $s=0$, thereby obtaining

$$\frac{1}{2\pi i}\int_{(c')}\frac{x^{-s}\mathrm{d}s}{2\sin\left(\frac{\pi s}{2}\right)}=\frac{1}{\pi(1+x^2)}-\frac{1}{\pi}. \tag{4.56}$$

Hence, from (4.55) and (4.56), we obtain

$$I=\frac{-x^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{\lambda(n)}{n(x^2+n^2)}+R_{1/2}+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}R_{\rho_m}. \tag{4.57}$$

Now (2.11) follows from equating the expressions for I in (4.53) and (4.57) (while using (4.54)), and simplifying.

4.5. **Ramanujan-Guinand type identity for $\lambda(n)$.** Before proving Theorem 2.7, we begin with a lemma.

Lemma 4.3. *For $c=\mathrm{Re}(s)<1/2$, we have*

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(\frac{1-s}{2})}t^{-s}\mathrm{d}s=\frac{\exp\left(-\frac{1}{8t^2}\right)}{4\sqrt{2}\pi t^2}\left(K_{\frac{1}{4}}\left(\frac{1}{8t^2}\right)+K_{\frac{3}{4}}\left(\frac{1}{8t^2}\right)\right).$$

Proof. From [51, p. 666, formula, **8.4.23.4**], for $\mathrm{Re}(s)<-|\mathrm{Re}(\nu)|$,

$$\int_0^{\infty}x^{s-1}e^{-1/(2x)}K_{\nu}\left(\frac{1}{2x}\right)\mathrm{d}x=\frac{\sqrt{\pi}\Gamma(\nu-s)\Gamma(-\nu-s)}{\Gamma\left(\frac{1}{2}-s\right)}.$$

Employ the change of variable $x=4t^2$ followed by the replacement of s by $s/2$ to obtain, for $\mathrm{Re}(s)<-2|\mathrm{Re}(\nu)|$,

$$\int_0^{\infty}t^{s-1}\exp\left(-\frac{1}{8t^2}\right)K_{\nu}\left(\frac{1}{8t^2}\right)\mathrm{d}t=\frac{\sqrt{\pi}\Gamma\left(\nu-\frac{s}{2}\right)\Gamma\left(-\nu-\frac{s}{2}\right)}{2^{s+1}\Gamma\left(\frac{1-s}{2}\right)}.$$

Now use the above equation, once with $\nu=1/4$, and then with $\nu=3/4$, and add the resulting equations to see that for $\mathrm{Re}(s)<-3/2$,

$$\int_0^{\infty}t^{s-1}\exp\left(-\frac{1}{8t^2}\right)\left(K_{\frac{1}{4}}\left(\frac{1}{8t^2}\right)+K_{\frac{3}{4}}\left(\frac{1}{8t^2}\right)\right)\mathrm{d}t=\frac{\sqrt{\pi}}{2^{s+1}}\frac{\Gamma\left(\frac{1}{4}-\frac{s}{2}\right)\Gamma\left(-\frac{1}{4}-\frac{s}{2}\right)+\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(-\frac{3}{4}-\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}.$$

Now replace s by $s-2$, divide both sides by $4\sqrt{2}\pi$, and then use the Mellin inversion (which is valid since the integral converges absolutely for $\mathrm{Re}(s)<1/2$), to obtain the result. □

Proof of Theorem 2.7. We evaluate

$$I(x):=\frac{1}{2\pi i}\int_{(3/2)}\frac{\pi^{-s}\zeta(2s)\Gamma(s)}{\pi^{-s/2}\zeta(s)\Gamma(s/2)}x^{-s}ds$$

in two different ways. On one hand, use (1.6) to get

$$I(x)=\sum_{n=1}^{\infty}\frac{\lambda(n)}{2\pi i}\int_{(3/2)}\frac{\Gamma(s)}{\Gamma(s/2)}(\sqrt{\pi}nx)^{-s}ds=\frac{x}{2}\sum_{n=1}^{\infty}n\lambda(n)e^{-\pi(xn)^2/4}, \tag{4.58}$$

using the duplication formula for the gamma function. On the other hand, we can shift the line of integration to $\mathrm{Re}(s) = -1/2$, and evaluate $I(x)$ using the residue theorem. As done in the proof of Theorem 2.1, there exists an infinite sequence $\{T_n\}_{n=0}^{\infty}$ such that

$$\lim_{T_n\to\infty}\int_{-1/2\pm iT_n}^{3/2\pm iT_n}\frac{\pi^{-(1-2s)/2}\zeta(1-2s)\Gamma(\frac{1-2s}{2})}{\pi^{-(1-s)/2}\zeta(1-s)\Gamma(\frac{1-s}{2})}x^{-s}ds = 0.$$

Hence invoking the residue theorem, using the symmetric form of the functional equation of $\zeta(s)$, considering the contributions of the poles at $1/2$ and at the non-trivial zeros ρ_m of $\zeta(s)$, and using (1.10) in the second step below, we find that

$$\begin{aligned} I(x) &= \frac{1}{2\pi i}\int_{(-1/2)}\frac{\pi^{-(1-2s)/2}\zeta(1-2s)\Gamma(\frac{1-2s}{2})}{\pi^{-(1-s)/2}\zeta(1-s)\Gamma(\frac{1-s}{2})}x^{-s}ds + R(x) \\ &= \sum_{n=1}^{\infty}\frac{c(n)}{n}\frac{1}{2\pi i}\int_{(-1/2)}\frac{\Gamma(\frac{1-2s}{2})}{\Gamma(\frac{1-s}{2})}\left(\frac{x}{\sqrt{\pi}n}\right)^{-s}ds + R(x), \end{aligned} \tag{4.59}$$

where

$$R(x) = \frac{\pi^{1/4}}{2\sqrt{x}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)} + \lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)\Gamma(\rho_m)}{\zeta'(\rho_m)\Gamma(\frac{\rho_m}{2})}(\sqrt{\pi}x)^{-\rho_m}.$$

The result now follows from invoking Lemma 4.3 in (4.59) and then equating the resulting right-hand side with that of (4.58). □

Remark 10. *Theorem 2.7 can also be proved from the Voronoï summation formula for $\lambda(n)$, that is, Theorem 2.1, by choosing $\phi(y) = \frac{1}{2}xye^{-\pi x^2y^2/4}$.*

5. Proofs of the Voronoi summation formula for $d^2(n)$

We begin with some lemmas.

Lemma 5.1. *For $\mathrm{Re}(s) > 1$, let*

$$\frac{\zeta^4(s)}{\zeta(2s-1)} = \sum_{n=1}^{\infty}\frac{b(n)}{n^s}.$$

Then $b(1) = 1$, and at prime powers $p^k, k \geq 0$, $b(p^k)$ is given by (1.14).

Proof. For $\mathrm{Re}(s) > 1$, $\zeta^4(s) = \sum_{n=1}^{\infty} d_4(n)n^{-s}$, where $d_4(n)$ is the number of ways in which n can be written as a product of 4 positive integers. Also, $1/\zeta(2s-1) = \sum_{n=1}^{\infty} a(n)n^{-s}$, where

$$a(n) = \begin{cases} m\mu(m), & \text{if } n = m^2, \\ 0, & \text{otherwise.} \end{cases}$$

and $\mathrm{Re}(s) > 1$. Since these two Dirichlet series converge absolutely in $\mathrm{Re}\, s > 1$, so does that of $b(n)$, and we have $b(n) = (a * d_4)(n)$. Since both a and d_4 are multiplicative, so is $b(n)$. Hence, $b(1) = 1$, and b is completely determined from its values at prime powers. We now evaluate $d_4(p^k), k \geq 0$. Let $p^k = a_1a_2a_3a_4$. Then $a_i = p^{x_i}, x_i \geq 0$, for $1 \leq i \leq 4$, so that $k = x_1 + x_2 + x_3 + x_4$. Since the number of weak compositions[6] of k into exactly r parts is given by $\binom{k+r-1}{r-1}$, it follows that $d_4(p^k) = \binom{k+3}{3}$. Thus,

$$b(p^k) = \sum_{d|p^k} a(d)d_4(p^k/d) = d_4(p^k) + p\mu(p)d_4(p^{k-2}) = \binom{k+3}{3} - p\binom{k+1}{3}.$$

□

[6] A weak composition of m is a composition of m in which 0 is allowed to be a part.

Lemma 5.2. *For* $0<\mathrm{Re}(w)<1/8$*,*

$$\int_0^\infty y^{w-1}\left(\frac{2}{\pi}K_0\big(4y^{1/4}\big)-Y_0\big(4y^{1/4}\big)\right)\mathrm{d}y=\frac{\Gamma^2(w)}{\Gamma^2(\frac{1}{2}-w)},\tag{5.1}$$

and the integral converges absolutely in this range.

Proof. This follows from the standard evaluations [46, p. 115, formula 11.1; p. 93, formula 10.2], namely, for $\mathrm{Re}(s)>\pm\,\mathrm{Re}(z)$ and $a>0$,

$$\int_0^\infty x^{s-1}K_z(ax)\mathrm{d}x=2^{s-2}a^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right),$$

and, for $\pm\mathrm{Re}(z)<\mathrm{Re}(s)<\frac{3}{2}$,

$$\int_0^\infty x^{s-1}Y_z(ax)\mathrm{d}x=-\frac{1}{\pi}2^{s-1}a^{-s}\cos\left(\frac{1}{2}\pi(s-z)\right)\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right).$$

Now $K_0(x)\ll x^{-1/2}$ and $Y_0(x)\ll x^{-1/2}$ as $x\to\infty$ and for any $\epsilon>0$ we have $K_0(x)\ll x^{-\epsilon}$ and $Y_0(x)\ll x^{-\epsilon}$ as $x\to 0^+$. Thus, the integral in (5.1) converges absolutely for $0<\mathrm{Re}(w)<1/8$. □

Lemma 5.3. *Define* $g(x)$ *to be*

$$g(x):=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}\Phi(1-2w)x^{-w}\mathrm{d}w,\tag{5.2}$$

where $\Phi(s)$ *is a function satisfying* (2.13) *and* $-\frac{1}{8}-\frac{\delta}{4}<c=\mathrm{Re}(s)<\frac{1}{8}$*. Then, with* ϕ *defined in the statement of Theorem* 2.8*, we have*

$$g(x)=\int_0^\infty\frac{\phi(t)}{t\sqrt{2\pi x}}\cos\left(\frac{4}{t^{1/2}x^{1/4}}\right)\mathrm{d}t.\tag{5.3}$$

Proof. With $w=\sigma+it$, Stirling's formula (3.1) implies

$$\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}\ll t^{\frac{3}{2}-4\sigma},\tag{5.4}$$

and hence, with the help of (2.13),

$$\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}\Phi(1-2w)\ll t^{-3/2-\delta-4\sigma}$$

as $t\to\infty$. Thus $g(x)$ converges when $c>-3/8-\delta/4$, and converges absolutely when $c>-1/8-\delta/4$. Let c be such that[7] $-1/8-\delta/4<c<1/8$.

From (5.4), it is seen that the integral

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}x^{-w}\mathrm{d}w$$

is absolutely convergent for $c>5/8$. But the line of integration in (5.2) is such that $c=\mathrm{Re}(w)<1/8$. The condition $\mathrm{Re}(w)>5/8$ is necessary to satisfy the hypotheses of Parseval's formula, that is, Theorem 3.1 which is what we intend to apply to the integral in (5.2). Hence we first shift the line of integration to $\mathrm{Re}(w)=5/7(>5/8)$. Showing that the integrals along the horizontal segments approach zero as

[7]We enforce the restriction $c<1/8$ since this condition appears in the evaluation of the integral in (5.16) while using Hardy's result for simplifying the kernel in the proof of Theorem 2.8.

the height of the rectangular contour tends to $\pm\infty$ and considering the contribution of the pole of the integrand at $w = 1/2$, by Cauchy's residue theorem, we get

$$\begin{aligned} g(x) &= \frac{1}{2\pi i}\int_{(5/7)}\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}\Phi(1-2w)x^{-w}\mathrm{d}w + x^{-1/2}\frac{\sqrt{\pi}\Phi(0)}{\Gamma(1/4)\Gamma(3/4)} \\ &= \frac{1}{2\pi i}\int_{(-3/7)}\frac{\Gamma(\frac{s}{2})\Gamma\left(\frac{1+s}{2}\right)x^{-\frac{(1-s)}{2}}}{2\Gamma\left(\frac{1}{4}-\frac{s}{2}\right)\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)}\Phi(s)\mathrm{d}s + x^{-1/2}\frac{\sqrt{\pi}\Phi(0)}{\Gamma(1/4)\Gamma(3/4)}, \end{aligned} \tag{5.5}$$

where, in the last step, we made a change of variable $w = (1-s)/2$. To apply Parseval's formula, we now show that the hypotheses of Theorem 3.1 are satisfied. Let $F(s) = \frac{\Gamma(\frac{1-s}{2})\Gamma\left(1-\frac{s}{2}\right)x^{-s/2}}{2\Gamma\left(\frac{s}{2}-\frac{1}{4}\right)\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)}$ and $G(s) = \Phi(s)$. First of all, the line of integration $\mathrm{Re}(s) = -3/7$ lies in the common strip of analyticity of $F(1-s)$ and $G(s)$. Next, (5.4) implies $F(10/7 - it) \in L(-\infty, \infty)$. Moreover, $\int_0^\infty t^{-10/7}|\phi(t)|\mathrm{d}t < \infty$ since $\phi(t) = O(t^{1-\epsilon})$ as $t \to 0^+$ and $\phi(t) = O(t^{-2+\epsilon})$ as $t \to \infty$. The latter, in turn, follows from the fact that $\Phi(s)$ is holomorphic in $-1 < \mathrm{Re}(s) < 2$. Thus, invoking Theorem 3.1, we see that

$$\frac{x^{-1/2}}{4\pi i}\int_{(-3/7)}\frac{\Gamma(\frac{s}{2})\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{s}{2}\right)\Gamma\left(\frac{3}{4}-\frac{s}{2}\right)}\Phi(s)x^{s/2}\mathrm{d}s = \int_0^\infty P_x(t)\phi(t)\mathrm{d}t, \tag{5.6}$$

where

$$P_x(t) := \frac{1}{2\pi i}\int_{(10/7)}F(s)t^{-s}\mathrm{d}s = \frac{1}{2\pi i}\int_{(5/7)}\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}(t^2x)^{-w}\mathrm{d}w,$$

where we made the change of variable $s = 2w$.

We now evaluate $P_x(t)$. To express it in terms of well-known functions, we need to shift the line of integration to $3/8 < \lambda = \mathrm{Re}(w) < 1/2$. The residue theorem then gives

$$P_x(t) = \frac{1}{2\pi i}\int_{(\lambda)}\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}(t^2x)^{-w}\mathrm{d}w - \frac{\sqrt{\pi}x^{-1/2}}{t\Gamma(1/4)\Gamma(3/4)}, \tag{5.7}$$

Next, we show

$$\frac{1}{2\pi i}\int_{(\lambda)}\frac{\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}x^{-w}\mathrm{d}w = \frac{\cos\left(4x^{-1/4}\right)}{\sqrt{2\pi x}}. \tag{5.8}$$

Replacing w and t in (4.30) by $2-4w$ and $4x^{-1/4}$ respectively, we see that for $3/8 < \lambda = \mathrm{Re}(w) < 1/2$,

$$\frac{1}{2\pi i}\int_{(\lambda)} -\Gamma(2-4w)\cos(2\pi w)2^{8w-2}x^{-w}\mathrm{d}w = \frac{\cos\left(4x^{-1/4}\right)}{\sqrt{x}}. \tag{5.9}$$

Using the Gauss multiplication formula [59, p. 52]

$$\prod_{k=1}^{m}\Gamma\left(z+\frac{k-1}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)}m^{\frac{1}{2}-mz}\Gamma(mz)$$

with $m = 4$ and $z = 1/2 - w$, and the reflection formula for the Gamma function, the integrand in (5.9) simplifies to $\frac{\sqrt{2\pi}\Gamma(\frac{1}{2}-w)\Gamma(1-w)}{\Gamma(w-\frac{1}{4})\Gamma(w+\frac{1}{4})}$, whence (5.8) follows. Hence from (5.7) and (5.8),

$$P_x(t) = \frac{\cos\left(4t^{-1/2}x^{-1/4}\right)}{t\sqrt{2\pi x}} - \frac{\sqrt{\pi}x^{-1/2}}{t\Gamma(1/4)\Gamma(3/4)},$$

which, along with (5.5) and (5.6), leads to

$$g(x) = \int_0^\infty \phi(t)\left(\frac{\cos(4t^{-1/2}x^{-1/4})}{t\sqrt{2\pi x}} - \frac{x^{-1/2}}{t}\frac{\sqrt{\pi}}{\Gamma(1/4)\Gamma(3/4)}\right)\mathrm{d}t + \frac{\Phi(0)\sqrt{\pi}x^{-1/2}}{\Gamma(1/4)\Gamma(3/4)}$$

$$= \int_0^\infty \phi(t)\left(\frac{\cos(4t^{-1/2}x^{-1/4})}{t\sqrt{2\pi x}} - \frac{x^{-1/2}}{t}\frac{\sqrt{\pi}}{\Gamma(1/4)\Gamma(3/4)}\right)\mathrm{d}t + \frac{\sqrt{\pi}x^{-1/2}}{\Gamma(1/4)\Gamma(3/4)}\int_0^\infty \frac{\phi(t)}{t}\mathrm{d}t$$
$$= \int_0^\infty \phi(t)\frac{\cos(4t^{-1/2}x^{-1/4})}{t\sqrt{2\pi x}}\mathrm{d}t,$$

where in the second step we used the fact that $\Phi(s) = \int_0^\infty t^{s-1}\phi(t)\mathrm{d}t$ for any s such that $-1 < \mathrm{Re}(s) < 2$. This completes the proof.

□

We are now ready to prove the Voronoï summation formula for $d^2(n)$.

Proof of Theorem 2.8. Let τ be a number satisfying $0 < \tau < \delta/2$, where $\delta > 0$ is the number occurring in (2.13). Define

$$I := \frac{1}{2\pi i}\int_{(1+\tau)} \frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s.$$

Using the fact that for $\mathrm{Re}(s) > 1$, $\zeta^4(s)/\zeta(2s) = \sum_{n=1}^\infty d^2(n)n^{-s}$ (which follows by letting $a = b = 0$ in (2.20)), and proceeding along the similar lines as (4.11), we obtain

$$I = \sum_{n=1}^\infty d^2(n)\phi(n). \tag{5.10}$$

We now to shift the line of integration to $-\tau = \mathrm{Re}\, s < 0$ by constructing the contour $[1+\tau-iT_n, 1+\tau+iT_n, -\tau+iT_n, -\tau-iT_n]$, where T_n is a number which belongs to the sequence $\{T_m\}_{m=1}^\infty$ constructed in the proof of Theorem 2.1. We first study the behavior of the integrals along the horizontal segments $-\tau+iT_n$ to $1+\tau+iT_n$ for large values of T_n. From (2.13), (4.15), (4.17) and (4.18),

$$\begin{aligned}\left|\int_{1+\tau+iT_n}^{-\tau+iT_n} \frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s\right| &= \left|\int_{-\tau}^{1+\tau} \frac{\zeta^4(u+iT_n)}{\zeta(2u+2iT_n)}\Phi(u+iT_n)\mathrm{d}u\right| \\ &\leq \left|\int_{-\tau}^{1/2} \frac{\zeta^4(u+iT_n)}{\zeta(2u+2iT_n)}\Phi(u+iT_n)\mathrm{d}u\right| + \left|\int_{1/2}^{1+\tau} \frac{\zeta^4(u+iT_n)}{\zeta(2u+2iT_n)}\Phi(u+iT_n)\mathrm{d}u\right| \\ &\leq \int_{-\tau}^{1/2} T_n^3T_n^\epsilon T_n^{-3-\delta}\mathrm{d}u + \int_{1/2}^{1+\tau} T_n^1T_n^\epsilon T_n^{-3-\delta}du.\end{aligned}$$

Now choose $\epsilon = \delta/2$ so that $\lim_{T_n\to\infty}\int_{-\tau}^{1/2} T_n^3T_n^\epsilon T_n^{-3-\delta}\mathrm{d}u = 0$. Also, $\lim_{T_n\to\infty}\int_{1/2}^{1+\tau} T_n^1T_n^\epsilon T_n^{-3-\delta}du = 0$. In conclusion, $\lim_{T_n\to\infty}\int_{1+\tau+iT_n}^{-\tau+iT_n}\frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s = 0$. Similarly one can show that the integral along the horizontal segment $[-\tau-iT_n, 1+\tau-iT_n]$ approaches zero as $T_n\to\infty$. Therefore, by the residue theorem,

$$I = \frac{1}{2\pi i}\int_{(-\tau)} \frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s + R_1 + \lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n} R_{\frac{\rho_m}{2}}, \tag{5.11}$$

where R_a denotes the residue of the integrand $\zeta^4(s)\Phi(s)/\zeta(2s)$ at $s = a$.

$$\begin{aligned}R_1 &= \frac{1}{6}\lim_{s\to1}\frac{d^3}{ds^3}(s-1)^4\frac{\zeta^4(s)}{\zeta(2s)}\Phi(s) \\ &= \frac{1}{\pi^8}\Big[\Phi(1)\big(24\gamma^3\pi^6 - 72\gamma\pi^6\gamma_1 + 12\pi^6\gamma_2 - 432\gamma^2\pi^4\zeta'(2) + 288\pi^4\gamma_1\zeta'(2) + 3456\gamma\pi^2(\zeta'(2))^2 - 10368(\zeta'(2))^3 \\ &\quad - 288\gamma\pi^4\zeta''(2) + 1728\pi^2\zeta'(2)\zeta''(2) - 48\pi^4\zeta'''(2)\big) \\ &\quad + \Phi'(1)\left(36\gamma^2\pi^6 - 24\pi^6\gamma_1 - 288\gamma\pi^4\zeta'(2) + 864\pi^2(\zeta'(2))^2 - 72\pi^4\zeta''(2)\right) \\ &\quad + \Phi''(1)\left(12\gamma\pi^6 - 36\pi^4\zeta'(2)\right) + \pi^6\Phi'''(1)\Big]\end{aligned}$$

$$= \int_0^\infty (A_0 + A_1 \log x + A_2 \log^2 x + A_3 \log^3 x)\phi(x)\mathrm{d}x, \tag{5.12}$$

where A_0, A_1, A_2 and A_3 are defined in (2.15). The last step is now justified. The Mellin inversion theorem [41, p. 341, Theorem 1] implies $\Phi(s) = \int_0^\infty x^{s-1}\phi(x)\mathrm{d}x$ in $-1 < \mathrm{Re}(s) < 2$. Owing to the fact that this integral is analytic in the given vertifcal strip, we can also differentiate this equation under the integral sign with respect to s. In particular, this gives

$$\Phi(1) = \int_0^\infty \phi(x)\mathrm{d}x, \qquad \Phi'(1) = \int_0^\infty \phi(x)\log(x)\mathrm{d}x,$$

$$\Phi''(1) = \int_0^\infty \phi(x)\log^2(x)\mathrm{d}x, \qquad \Phi'''(1) = \int_0^\infty \phi(x)\log^3(x)\mathrm{d}x,$$

which justifies the last step. Also, if ρ_m is the mth non-trivial zero of $\zeta(s)$, then

$$R_{\frac{\rho_m}{2}} = \lim_{s\to\frac{\rho_m}{2}} \frac{\zeta^4(s)}{\zeta(2s)} \int_0^\infty \phi(x)x^{s-1}dx = \frac{\zeta^4(\frac{\rho_m}{2})}{2\zeta'(\rho_m)} \int_0^\infty \phi(x)x^{\frac{\rho_m}{2}-1}dx. \tag{5.13}$$

It remains to evaluate the integral $\frac{1}{2\pi i}\int_{(-\tau)} \zeta^4(s)\Phi(s)/\zeta(2s)\mathrm{d}s$. To that end, using (4.13), we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{(-\tau)} \frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s &= \frac{\pi^{-3/2}}{2\pi i}\int_{(-\tau)} \frac{\zeta^4(1-s)}{\zeta(1-2s)}\frac{\Gamma^4(\frac{1-s}{2})}{\Gamma^4(\frac{s}{2})}\frac{\Gamma(s)}{\Gamma(\frac12 - s)}\Phi(s)\pi^{2s}\mathrm{d}s \\
&= \frac{\pi^{-3/2}}{2\pi i}\int_{(1+\tau)} \frac{\zeta^4(s)}{\zeta(2s-1)}\frac{\Gamma^4(\frac{s}{2})}{\Gamma^4(\frac{1-s}{2})}\frac{\Gamma(1-s)}{\Gamma(s-\frac12)}\Phi(1-s)\pi^{2(1-s)}\mathrm{d}s \\
&= \frac{\pi^{1/2}}{2\pi i}\sum_{n=1}^\infty b(n)\int_{(1+\tau)} \frac{\Gamma^4(\frac{s}{2})}{\Gamma^4(\frac{1-s}{2})}\frac{\Gamma(1-s)}{\Gamma(s-\frac12)}\Phi(1-s)(\pi^2 n)^{-s}\mathrm{d}s,
\end{aligned} \tag{5.14}$$

where, in the last step, we invoked Lemma (5.1) and interchanged the order of summation and integration using absolute and uniform convergence. Indeed, this follows from the fact that $\mathrm{Re}(s) > 1$ and from (3.1), (2.13), and the choice of τ, namely, $\tau < \delta/2$. Now let

$$M(t) := \frac{1}{2\pi i}\int_{(1+\tau)} \frac{\Gamma^4(\frac{s}{2})}{\Gamma^4(\frac{1-s}{2})}\frac{\Gamma(1-s)}{\Gamma(s-\frac12)}\Phi(1-s)t^{-s}\mathrm{d}s. \tag{5.15}$$

By the change of variable $s = 2w$ and an application of the duplication formula, we have

$$M(t) = \frac{2^{5/2}}{2\pi i}\int_{(\frac{1+\tau}{2})} \frac{\Gamma^4(w)}{\Gamma^4\left(\frac12 - w\right)}\frac{\Gamma\left(\frac12 - w\right)}{\Gamma\left(w - \frac14\right)}\frac{\Gamma(1-w)}{\Gamma\left(w+\frac14\right)}\Phi(1-2w)(4t)^{-2w}\mathrm{d}w. \tag{5.16}$$

The main task now is to express $M(t)$ as a (multiple) integral whose integrand consists of ϕ and well-known functions. Even though it looks like the extension of the usual version of Parseval's formula for three functions would be applicable in this situation, one can see that it is, in fact, difficult to apply. This is where a variant of such a formula due to G. H. Hardy comes in very handy.

By extending Theorems A and C in [29] to p functions, Hardy derived a result [29, p. 91] in the same paper whose special case for $p = 3$ is

$$\frac{1}{2\pi i}\int_{(c)} f_1(w)f_2(w)f_3(w)t^{-w}\mathrm{d}w = \int_0^\infty\int_0^\infty \phi_1(x)\phi_2(y)\phi_3\left(\frac{t}{xy}\right)\frac{\mathrm{d}x\mathrm{d}y}{xy}.$$

The conditions on $f_j, \phi_j, 1 \le j \le 3$, which make the above result valid are extensions of the conditions given for the two functions in Theorem A of [29]. We use this result with $\phi_1(x) = \phi_2(x) = \frac{2}{\pi}K_0\left(4x^{1/4}\right) - Y_0\left(4x^{1/4}\right)$ and $\phi_3(x) = g(x)$, where g is the function in (5.3), $\alpha = \epsilon, \beta = -1/8 - \delta/4, \gamma = 1/2$ and[8]

[8]We changed Hardy's notation δ to δ' so as to not get confused with our delta stemming from (2.13).

$\delta' = 1/8$, and then employ Lemmas 5.2 and 5.3. Upon simplification, this gives

$$M(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^\infty\left(\frac{2}{\pi}K_0\big(4x^{1/4}\big)-Y_0\big(4x^{1/4}\big)\right)\left(\frac{2}{\pi}K_0\big(4y^{1/4}\big)-Y_0\big(4y^{1/4}\big)\right)$$
$$\times\int_0^\infty\frac{\phi(z)}{tz}\cos\left(\frac{2(xy)^{1/4}}{\sqrt{tz}}\right)\frac{\mathrm{d}z\mathrm{d}x\mathrm{d}y}{\sqrt{xy}}$$
$$=\frac{4}{\sqrt{\pi}}\int_0^\infty\int_0^\infty\int_0^\infty\left(\frac{2}{\pi}K_0\big(4\sqrt{x}\big)-Y_0\big(4\sqrt{x}\big)\right)\left(\frac{2}{\pi}K_0\big(4\sqrt{y}\big)-Y_0\big(4\sqrt{y}\big)\right)\frac{\phi(z)}{tz}\cos\left(\frac{2\sqrt{xy}}{\sqrt{tz}}\right)\mathrm{d}z\mathrm{d}x\mathrm{d}y, \tag{5.17}$$

where, in the last step, we replaced x and y by x^2 and y^2. Therefore, from (5.14), (5.15) and (5.17),

$$\frac{1}{2\pi i}\int_{(-\tau)}\frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s = \frac{4}{\pi^2}\sum_{n=1}^\infty\frac{b(n)}{n}\int_0^\infty\int_0^\infty\int_0^\infty\left(\frac{2}{\pi}K_0\big(4\sqrt{x}\big)-Y_0\big(4\sqrt{x}\big)\right)$$
$$\times\left(\frac{2}{\pi}K_0\big(4\sqrt{y}\big)-Y_0\big(4\sqrt{y}\big)\right)\frac{\phi(z)}{z}\cos\left(\frac{2\sqrt{xy}}{\pi\sqrt{nz}}\right)\mathrm{d}z\mathrm{d}x\mathrm{d}y. \tag{5.18}$$

Finally, from (5.10), (5.11), (5.12), (5.13) and (5.18), we arrive at (2.14).

□

6. Proofs of the results on $\sigma_a(n)\sigma_b(n)$

6.1. Cohen type identity for $\sigma_a(n)\sigma_b(n)$. Before we embark upon the proof of Theorem 2.11, we need the following two lemmas.

Lemma 6.1. *Let $x \in \mathbb{C}, |x| \neq 1$, and $-1 < \mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(a-b), \mathrm{Re}(a+b) < 1$. Define $I(x)$ by*

$$I(x) = \frac{1}{2\pi i}\int_{(c)}\frac{\sec\left(\frac{\pi}{2}s\right)\sec(\frac{\pi}{2}(s-a))\sec(\frac{\pi}{2}(s-b))\sec(\frac{\pi}{2}(s-a-b))}{\sec(\frac{\pi}{2}(2s-a-b-1))}x^{-s}\mathrm{d}s,$$

where $\max\{-1,-1+\mathrm{Re}(a),-1+\mathrm{Re}(b),-1+\mathrm{Re}(a+b)\} < c = \mathrm{Re}(s) < \min\{1,1+\mathrm{Re}(a),1+\mathrm{Re}(b),1+\mathrm{Re}(a+b)\}$. *Then*

$$I(x) = \frac{2}{\pi}\mathrm{cosec}\left(\frac{\pi a}{2}\right)\mathrm{cosec}\left(\frac{\pi b}{2}\right)\frac{x(x^{-a}-1)(x^{-b}-1)}{x^2-1}.$$

Proof. Assume $|x| < 1$. Construct a rectangular contour $[c-iT, c+iT, -N+iT, -N-iT, c-iT]$, where

$$\max\{-2m-3,-2m-3+a,-2m-3+b,-2m-3+a+b\} < N < \min\{-2m-1,-2m-1+a,-2m-1+b,-2m-1+a+b\}$$

for $m \geq 0$. The integrals over the horizontal segments tend to zero as $T \to \infty$ as can be seen by the exponential decay of the integrand, which, in turn, can be inferred by first rewriting the integrand using $\Gamma\left(\frac{1}{2}+w\right)\Gamma\left(\frac{1}{2}-w\right) = \pi/\cos(\pi w)$ and then using Stirling's formula (3.1). Hence by the residue theorem,

$$I(x) = \frac{1}{2\pi i}\int_{-N-i\infty}^{-N+i\infty}\frac{\sec\left(\frac{\pi}{2}s\right)\sec(\frac{\pi}{2}(s-a))\sec(\frac{\pi}{2}(s-b))\sec(\frac{\pi}{2}(s-a-b))}{\sec(\frac{\pi}{2}(2s-a-b-1))}x^{-s}\mathrm{d}s$$
$$+\sum_{n=0}^{m}\left(R_{-2n-1}+R_{-2n-1+a}+R_{-2n-1+b}+R_{-2n-1+a+b}\right),$$

where $R_{-2n-1+u} = \pm 2x^{1-u}\mathrm{cosec}\left(\frac{\pi a}{2}\right)\mathrm{cosec}\left(\frac{\pi b}{2}\right)x^{2n}/\pi$, where u takes the values $0, a, b$ or $a+b$, and we take minus sign when $u = 0$ or $a+b$ and plus sign when $u = a$ or b.

We now let $N \to \infty$ in the above equation. It is not difficult to see that the integral on the right-hand side tends to zero. Indeed, (3.1) and the fact that $|x| < 1$ shows that the integral approaches zero in the limit. This leads to

$$I(x) = \sum_{n=0}^{\infty}\left(R_{-2n-1}+R_{-2n-1+a}+R_{-2n-1+b}+R_{-2n-1+a+b}\right)$$

$$= \frac{2x}{\pi}\operatorname{cosec}\left(\frac{\pi a}{2}\right)\operatorname{cosec}\left(\frac{\pi b}{2}\right)\left(1 - x^{-a} - x^{-b} + x^{-a-b}\right)\sum_{n=0}^{\infty} x^{2n}$$

$$= \frac{2}{\pi}\operatorname{cosec}\left(\frac{\pi a}{2}\right)\operatorname{cosec}\left(\frac{\pi b}{2}\right)\frac{x(x^{-a}-1)(x^{-b}-1)}{x^2-1}. \tag{6.1}$$

This proves the result for $|x| < 1$. Now if $|x| > 1$, we shift the line of integration to $+\infty$, and proceed along the similar lines as above. This leads to the same evaluation of $I(x)$ as in (6.1). □

To the best of our knowledge, the following evaluation of Meijer G-function (defined in (3.3)) seems to be new.

Lemma 6.2. *For* $c = \mathrm{Re}(s) > \max\{0, \mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(a+b)\}$,

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)\Gamma(s-a-b)}{\Gamma\left(s - \frac{a+b+1}{2}\right)\Gamma\left(s-\frac{a+b}{2}\right)} z^{-s}\mathrm{d}s$$

$$= G^{4,\,0}_{2,\,4}\left(\left.\begin{matrix}\frac{-a-b}{2}, \frac{-a-b-1}{2}\\ 0, -a, -b, -a-b\end{matrix}\right| z\right)$$

$$= \frac{1}{\sqrt{\pi}} z^{\frac{1-a-b}{2}}\left(K_{a-1}(\sqrt{z})K_b(\sqrt{z}) + K_{b-1}(\sqrt{z})K_a(\sqrt{z}) + \frac{(a+b-1)}{\sqrt{z}}K_a(\sqrt{z})K_b(\sqrt{z})\right). \tag{6.2}$$

Proof. The first equality follows from the definition of Meijer G-function. Now it is known [66] that

$$G^{4,\,0}_{2,\,4}\left(\left.\begin{matrix}A, A+\frac{1}{2}\\ B, C, 2A-C, 2A-B\end{matrix}\right| z\right) = \frac{2}{\sqrt{\pi}} z^A K_{B-C}(\sqrt{z})K_{B+C-2A}(\sqrt{z}). \tag{6.3}$$

Let $A = \frac{-a-b}{2}, B = 0$ and $C = -a$. Then the resulting Meijer G-function differs from the one in (6.2) only in one of the top parameters with the difference between them being 1. To address this issue, we use the identity [51, p. 621, Formula (37)], namely, for $n \le p-1$,

$$\frac{d}{dz}\left[z^{1-a_p}G^{m,\,n}_{p,\,q}\left(\left.\begin{matrix}a_1, \cdots, a_n, a_{n+1}, \cdots, a_p\\ b_1, \cdots, b_m, b_{m+1}, \cdots, b_q\end{matrix}\right| z\right)\right] = -z^{-a_p}G^{m,\,n}_{p,\,q}\left(\left.\begin{matrix}a_1, \cdots, a_n, a_{n+1}, \cdots, a_p - 1\\ b_1, \cdots, b_m, b_{m+1}, \cdots, b_q\end{matrix}\right| z\right)$$

Therefore letting $m = 4, n = 0, p = 2, q = 4$ and $a_p = (1-a-b)/2$ in the above identity, and using (6.3) in the second step, we see that

$$G^{4,\,0}_{2,\,4}\left(\left.\begin{matrix}\frac{-a-b}{2}, \frac{-a-b-1}{2}\\ 0, -a, -b, -a-b\end{matrix}\right| z\right)$$

$$= -z^{\frac{1-a-b}{2}}\frac{d}{dz}\left(z^{\frac{1+a+b}{2}}G^{4,\,0}_{2,\,4}\left(\left.\begin{matrix}\frac{-a-b}{2}, \frac{-a-b+1}{2}\\ 0, -a, -b, -a-b\end{matrix}\right| z\right)\right)$$

$$= -\frac{2}{\sqrt{\pi}} z^{\frac{1-a-b}{2}}\frac{d}{dz}\left(\sqrt{z}K_a(\sqrt{z})K_b(\sqrt{z})\right)$$

$$= \frac{1}{\sqrt{\pi}} z^{\frac{1-a-b}{2}}\left(K_{a-1}(\sqrt{z})K_b(\sqrt{z}) + K_{b-1}(\sqrt{z})K_a(\sqrt{z}) + \frac{(a+b-1)}{\sqrt{z}}K_a(\sqrt{z})K_b(\sqrt{z})\right),$$

where, in the last step, we used the standard formula for the differentiation of the K-Bessel function [26, p. 929, **8.486.11**], namely, $\frac{d}{dw}K_\nu(w) = -\frac{1}{2}\left(K_{\nu-1}(w) + K_{\nu+1}(w)\right)$. This completes the proof. □

Remark 11. *We note a related formula which appears in* [13, p. 647, Formula (55)]*:*

$$G^{4,\,0}_{2,\,4}\left(\left.\begin{matrix}\frac{1}{2}, 1\\ a+b, a-b, b-a, -a-b\end{matrix}\right| z\right) = \frac{2}{(a+b)\sqrt{\pi}}K_{2a}(\sqrt{z})K_{2b}(\sqrt{z})$$

$$+ \frac{\sqrt{z}}{(a^2-b^2)\sqrt{\pi}}\left(K_{2a-1}(\sqrt{z})K_{2b}(\sqrt{z}) - K_{2a}(\sqrt{z})K_{2b-1}(\sqrt{z})\right).$$

Armed with these results, we now ready to prove the Cohen-type identity for $\sigma_a(n)\sigma_b(n)$.

Proof of Theorem 2.11. We assume

$$\max\{-1,-1+\mathrm{Re}(a),-1+\mathrm{Re}(b),-1+\mathrm{Re}(a+b)\}<c<\min\left\{0,\mathrm{Re}(a),\mathrm{Re}(b),\mathrm{Re}(a+b),\mathrm{Re}\left(\tfrac{a+b+1}{2}\right)\right\}.$$

The above strip is of positive width because of the conditions on a and b given in the hypotheses. Let

$$S(a,b,x):=\sum_{n=1}^{\infty}\sigma_a(n)\sigma_b(n)\frac{x\left(x^{-a}-n^{-a}\right)\left(x^{-b}-n^{-b}\right)}{x^2-n^2}.$$

Then an application of $\sigma_s(n)=\sigma_{-s}(n)n^s$ in the first step and Lemma 6.1 in the second leads to

$$\begin{aligned}
S(a,b,x)&=\sum_{n=1}^{\infty}\frac{\sigma_{-a}(n)\sigma_{-b}(n)}{n}\frac{\frac{x}{n}\left(\left(\frac{x}{n}\right)^{-a}-1\right)\left(\left(\frac{x}{n}\right)^{-b}-1\right)}{\frac{x^2}{n^2}-1}\\
&=\frac{\pi}{2}\sin\left(\frac{\pi a}{2}\right)\sin\left(\frac{\pi b}{2}\right)\sum_{n=1}^{\infty}\frac{\sigma_{-a}(n)\sigma_{-b}(n)}{n}\\
&\quad\times\frac{1}{2\pi i}\int_{(c)}\frac{\sec\left(\frac{\pi}{2}s\right)\sec(\frac{\pi}{2}(s-a))\sec(\frac{\pi}{2}(s-b))\sec(\frac{\pi}{2}(s-a-b))}{\sec(\frac{\pi}{2}(2s-a-b-1))}\left(\frac{x}{n}\right)^{-s}\mathrm{d}s\\
&=\frac{\pi}{2}\sin\left(\frac{\pi a}{2}\right)\sin\left(\frac{\pi b}{2}\right)\\
&\quad\times\frac{1}{2\pi i}\int_{(c)}\sum_{n=1}^{\infty}\frac{\sigma_{-a}(n)\sigma_{-b}(n)}{n^{1-s}}\frac{\sec\left(\frac{\pi}{2}s\right)\sec(\frac{\pi}{2}(s-a))\sec(\frac{\pi}{2}(s-b))\sec(\frac{\pi}{2}(s-a-b))}{\sec(\frac{\pi}{2}(2s-a-b-1))}x^{-s}\mathrm{d}s\\
&=\frac{\pi}{2}\sin\left(\frac{\pi a}{2}\right)\sin\left(\frac{\pi b}{2}\right)\frac{1}{2\pi i}\int_{(c)}\frac{\zeta(1-s)\zeta(1-s+a)\zeta(1-s+b)\zeta(1-s+a+b)}{\zeta(2-2s+a+b)}\\
&\quad\times\frac{\sec\left(\frac{\pi}{2}s\right)\sec(\frac{\pi}{2}(s-a))\sec(\frac{\pi}{2}(s-b))\sec(\frac{\pi}{2}(s-a-b))}{\sec(\frac{\pi}{2}(2s-a-b-1))}x^{-s}\mathrm{d}s,
\end{aligned}$$

where we interchanged the order of summation and integration because of the uniform convergence of the associated series in the above strip, and where we used (2.20) in the last step.

Next, use (1.20) for each of the zeta functions occurring in the integrand of the above equation and simplify to obtain

$$\begin{aligned}
S(a,b,x)=2(2\pi)^{a+b}\sin\left(\frac{\pi a}{2}\right)\sin\left(\frac{\pi b}{2}\right)\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)\Gamma(s-a-b)}{\Gamma(2s-a-b-1)}\\
\times\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b-1)}(4\pi^2x)^{-s}\mathrm{d}s.
\end{aligned}$$

Now we wish to invoke (2.17). It necessitates shifting the line of integration from $\mathrm{Re}(s)=c$ to $\mathrm{Re}(s)=\eta$, where η is defined in (2.18). Clearly, Stirling's formula (3.1) implies that the integrals along the horizontal segments tend to zero as the height of the rectangular contour approaches ∞. The integrand has simple poles at $s=0,a,b,a+b$ (due to the gamma functions), at $s=1,1+a,1+b,1+a+b$ (due to the zeta functions), and at the non-trivial zeros of $\zeta(2s-a-b-1)$ at $s=(1+\rho_m+a+b)/2$, where ρ_m are the non-trivial zeros of $\zeta(s)$. (Here we have assumed the simplicity of the zeros.)

The residues at these poles can be easily calculated to be the ones given in (2.24) and (2.25). Therefore invoking the residue theorem, expressing the quotient of zeta functions by the left-hand side of (2.17) and interchanging the order of summation and integration (valid due to uniform convergence), we find that

$$S(a,b,x)=2(2\pi)^{a+b}\sin\left(\tfrac{\pi a}{2}\right)\sin\left(\tfrac{\pi b}{2}\right)\left\{\sum_{n=1}^{\infty}C_{a,b}(n)\frac{1}{2\pi i}\int_{(\eta)}\frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)\Gamma(s-a-b)}{\Gamma(2s-a-b-1)}\left(4\pi^2nx\right)^{-s}\mathrm{d}s\right.$$

$$-\left(\sum_{k=0}^{1}R_k(x)+R_{k+a}(x)+R_{k+b}(x)+R_{k+a+b}(x)+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}R_{\rho_m,a,b}(x)\right)\Bigg\}.$$

This results in (2.23) upon invoking Lemma 6.2.

□

Proof of Corollary 2.12. Divide both sides of (2.23) by ab and then let $a\to 0$ and $b\to 0$. Using the fact that $\sigma_s(n)n^{-\frac{s}{2}}=O\left(n^{\frac{1}{2}|\mathrm{Re}(s)|+\epsilon}\right)$, it is easily seen that the series on the left-hand side of (2.23) is uniformly convergent in any compact interval of $(-1,1)$, viewed as a function of the complex variable a or b. Hence we can interchange the order of limits and summation. The same can be done on the right-hand side as well. This leads to (2.12) upon simplification[9]. □

6.2. **Ramanujan-Guinand type identity for $\sigma_a(n)\sigma_b(n)$.** We prove Theorem 2.13 in this subsection. Let $b=c=2$ and then $\alpha=s-a/2-b/2,\mu=a/2,\nu=b/3$ in [50, p. 384, formula **2.16.33.1**] so that for $c=\mathrm{Re}(s)>\max\{0,\mathrm{Re}(a),\mathrm{Re}(b),\mathrm{Re}(a+b)\}$,

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s-a}{2})\Gamma(\frac{s-b}{2})\Gamma(\frac{s-a-b}{2})}{\Gamma(\frac{2s-a-b}{2})}x^{-s}\mathrm{d}s=8x^{(-a-b)/2}K_{a/2}(2x)K_{b/2}(2x).$$

Now let $c=\mathrm{Re}(s)>1+\max(0,\mathrm{Re}(a),\mathrm{Re}(b),\mathrm{Re}(a+b))$ and define

$$I(x):=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{s}{2})\zeta(s)\Gamma(\frac{s-a}{2})\zeta(s-a)\Gamma(\frac{s-b}{2})\zeta(s-b)\Gamma(\frac{s-a-b}{2})\zeta(s-a-b)}{\Gamma(\frac{2s-a-b}{2})\zeta(2s-a-b)}x^{-s}\mathrm{d}s. \tag{6.4}$$

Then, using (2.20), we have

$$I(x)=8x^{(-a-b)/2}\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^{(a+b)/2}}K_{a/2}(2nx)K_{b/2}(2nx).$$

On the other hand, invoking (1.21) in the first step, we get

$$I(x)=\frac{\pi^{-a-b-\frac{3}{2}}}{2\pi i}\int_{(c)}\frac{\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\Gamma\left(\frac{1-s+a}{2}\right)\zeta(1-s+a)\Gamma\left(\frac{1-s+b}{2}\right)\zeta(1-s+b)\Gamma\left(\frac{1-s+a+b}{2}\right)\zeta(1-s+a+b)}{\Gamma\left(\frac{1-2s+a+b}{2}\right)\zeta(1-2s+a+b)\pi^{-2s}x^{s}}\mathrm{d}s.$$

Now shift the line of integration to $c'=\mathrm{Re}(s)<\min(0,\mathrm{Re}(a),\mathrm{Re}(b),\mathrm{Re}((a+b)/2),\mathrm{Re}(a+b))$ so as to be able to use (2.17), consider the contributions of the poles at $(\rho_m+a+b)/2$, and at $k,k+a,k+b$ and $k+a+b$, where $k=0$ or 1, and invoke the residue theorem to get

$$\begin{aligned}I(x)&=\pi^{-a-b-\frac{3}{2}}\sum_{n=1}^{\infty}\frac{C_{-a,-b}(n)}{n}\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s+a}{2}\right)\Gamma\left(\frac{1-s+b}{2}\right)\Gamma\left(\frac{1-s+a+b}{2}\right)}{\Gamma\left(\frac{1-2s+a+b}{2}\right)}\left(\frac{x}{\pi^2 n}\right)^{-s}\mathrm{d}s+R(x)\\&=\frac{\pi^{-a-b-1}}{2^{\frac{a+b-3}{2}}}\sum_{n=1}^{\infty}\frac{C_{-a,-b}(n)}{n}\frac{1}{2\pi i}\int_{(\frac{c'}{2})}\frac{\Gamma\left(\frac{1}{2}-w\right)\Gamma\left(\frac{1+a}{2}-w\right)\Gamma\left(\frac{1+b}{2}-w\right)\Gamma\left(\frac{1+a+b}{2}-w\right)}{\Gamma\left(\frac{1+a+b}{4}-w\right)\Gamma\left(\frac{3+a+b}{4}-w\right)}\left(\frac{x^2}{4\pi^4n^2}\right)^{-w}\mathrm{d}w\\&\quad+R(x),\end{aligned}$$

where $R(x)$ is the sum of the residues in (2.28) and (2.29). In the last step, we used the duplication formula and then employed the change of variable $s=2w$. The integral in the last step can be easily seen to be the Meijer G-function claimed in (2.27), which completes the proof.

Proof of Corollary 2.14. Let $a=b=0$ in Theorem 2.13. We found using *Mathematica* that x times the residue $\tilde{R}_1(x)$ is a polynomial in $\log(x)$. Althought it can be explicitly written down, we avoid giving it here as it is quite complicated. □

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[9]The residue $R_0(x)$ in this case was calculated using *Mathematica*.

7. Proofs of Theorem 2.9 and Ramanujan's identity (1.9)

Proof of Theorem 2.9. Using the Mellin inversion formula, we get $\Phi(s)=\int_0^\infty \phi(x)x^{s-1}\mathrm{d}x$ for $\mathrm{Re}\, s>0$ and $\Phi(s)=\int_0^\infty (\phi(x)-k)x^{s-1}\mathrm{d}x$ for $-1<\mathrm{Re}\, s<0$, where k is the residue of $\Phi(s)$ at $s=0$.

Choose τ such that $0<\tau<\delta$, where δ is the number defined in (2.16), and define

$$I:=\frac{1}{2\pi i}\int_{(1+\tau)}\frac{\Phi(s)}{\zeta(s)}\mathrm{d}s. \tag{7.1}$$

Then, one one hand,

$$I=\frac{1}{2\pi i}\int_{(1+\tau)}\frac{\Phi(s)}{\zeta(s)}\mathrm{d}s=\sum_{n=1}^\infty \mu(n)\frac{1}{2\pi i}\int_{(1+\tau)}\Phi(s)n^{-s}\mathrm{d}s=\sum_{n=1}^\infty \mu(n)\phi(n).$$

If we now shift the line of integration to $\mathrm{Re}(s)=-\tau$, and choose the sequence $\{T_n\}$ as in Theorem 2.1 so that the integrals along the horizontal segments go to zero. Then by Cauchy's residue theorem, we have

$$I=\frac{1}{2\pi i}\int_{(-\tau)}\frac{\Phi(s)}{\zeta(s)}\mathrm{d}s+\mathrm{Res}_{s=0}\frac{\Phi(s)}{\zeta(s)}+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{1}{\zeta'(\rho_m)}\int_0^\infty \phi(x)x^{\rho_m-1}\mathrm{d}x. \tag{7.2}$$

From (7.1) and (7.2), we have

$$\sum_{n=1}^\infty \mu(n)\phi(n)=\frac{1}{2\pi i}\int_{(-\tau)}\frac{\Phi(s)}{\zeta(s)}\mathrm{d}s-2k+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{1}{\zeta'(\rho_m)}\int_0^\infty \phi(x)x^{\rho_m-1}\mathrm{d}x. \tag{7.3}$$

With a fair amount of calculation using Parseval's formula (3.2), one can see that

$$\frac{1}{2\pi i}\int_{(-\tau)}\frac{\Phi(s)}{\zeta(s)}\mathrm{d}s=4\sum_{n=1}^\infty\frac{\mu(n)}{n}\int_0^\infty(\phi(t)-k)\left(\frac{\cos^2(\pi/nt)-1}{t}\right)\mathrm{d}t, \tag{7.4}$$

which, along with (7.3), completes the proof. □

We prove Corollary 2.10 here. Let $\phi(x)=\frac{\sqrt{a}}{x}\tilde{\phi}(\frac{a}{x})$ in Theorem 2.9, where $\tilde{\phi}$ is chosen so that the hypotheses of Theorem 2.9 are satisfied, and, in addition, there is no residue of $\Phi(s)$ at $s=0$, that is, k in Theorem 2.9 is equal to 0. Then

$$\begin{aligned}&\sqrt{a}\sum_{n=1}^\infty\frac{\mu(n)}{n}\tilde{\phi}\left(\frac{a}{n}\right)+4\sum_{n=1}^\infty\frac{\mu(n)}{n}\int_0^\infty\frac{\sqrt{a}}{t^2}\tilde{\phi}\left(\frac{a}{t}\right)\sin^2\left(\frac{\pi}{nt}\right)\mathrm{d}t\\&=\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{1}{\zeta'(\rho_m)}\int_0^\infty\frac{\sqrt{a}}{x}\tilde{\phi}\left(\frac{a}{x}\right)x^{\rho_m-1}\mathrm{d}x.\end{aligned}$$

Employ change of variables $t\to a/t$ and $x\to a/x$ in the two integrals and use the elementary identity $2\sin^2(\theta)=1-\cos(2\theta)$ so that

$$\begin{aligned}&\sqrt{a}\sum_{n=1}^\infty\frac{\mu(n)}{n}\tilde{\phi}\left(\frac{a}{n}\right)+\frac{2}{\sqrt{a}}\sum_{n=1}^\infty\frac{\mu(n)}{n}\int_0^\infty\tilde{\phi}(t)\left(1-\cos\left(\frac{2\pi t}{an}\right)\right)\mathrm{d}t\\&=\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{a^{\rho_m-1/2}}{\zeta'(\rho_m)}\int_0^\infty\tilde{\phi}(x)\,x^{-\rho_m}\mathrm{d}x.\end{aligned}$$

From the hypotheses given at the beginning of (1.9), the integral $\frac{2}{\sqrt{\pi}}\int_0^\infty\tilde{\phi}(u)\cos(2ux)\mathrm{d}u$ exists; hence defining it to be $\tilde{\psi}(x)$ and then employing it and the prime number theorem in the form $\sum_{n=1}^\infty\mu(n)/n=0$, we are led to

$$\sqrt{\alpha}\sum_{n=1}^\infty\frac{\mu(n)}{n}\tilde{\phi}\left(\frac{\alpha}{n}\right)-\frac{\sqrt{\pi}}{\sqrt{\alpha}}\sum_{n=1}^\infty\frac{\mu(n)}{n}\tilde{\psi}\left(\frac{\pi}{\alpha n}\right)=\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\alpha^{\rho_m-1/2}}{\zeta'(\rho_m)}\Gamma\left(1-\rho_m\right)Z_1\left(1-\rho_m\right).$$

Now let $\beta = \pi/\alpha$ and use (1.8) to arrive at the first equality in (1.9). The second follows by swapping α and β in the first and using the second integral in (1.7).

8. Oscillations of Riesz sums

Throughout this section we assume RH and the simplicity of the zeros of $\zeta(s)$. The Gonek-Hejhal conjecture [25], [32] (see also [45, Equation (3)] states that for $k \in \mathbb{R}$,

$$\sum_{|\gamma_m| \leq T} \frac{1}{|\zeta'(\rho_m)|^{2k}} \asymp T(\log T)^{(k-1)^2}. \tag{8.1}$$

In particular, we have

$$\sum_{|\gamma_m| \leq T} \frac{1}{|\zeta'(\rho_m)|} \asymp T(\log T)^{1/4}.$$

The conjecture is still open but assuming RH, Heap, Li and Zhao [31] have proved the conjectured lower bound in (8.1) for all fractional $k \geq 0$, assuming RH and the simplicity of the zeros; in particular, for $k = 1/2$, we have

$$\sum_{|\gamma_m| \leq T} \frac{1}{|\zeta'(\rho_m)|} \gg T(\log T)^{1/4}. \tag{8.2}$$

The special case $k = 1/2$ of a recent result of Bui, Florea and Milinovich [12, Theorem 1.1] implies that for any $\delta > 0$,

$$\sum_{\gamma_m \in \mathcal{F}_T} \frac{1}{|\zeta'(\rho_m)|} \ll T^{1+\delta},$$

where

$$\mathcal{F}_T = \left\{ \gamma \in (T, 2T] : |\gamma - \gamma'| \gg \frac{1}{\log T} \text{ for any other ordinate } \gamma' \right\}.$$

Since there is no non-trivial zero of $\zeta(s)$ with $|\gamma_m| < 14$, if $\mathcal{F} = \sqcup_{k=0}^{\lfloor \log_2(T) \rfloor} \mathcal{F}_{T/2^k}$, then for any $\delta > 0$,

$$\sum_{\substack{|\gamma_m| \leq T \\ \gamma_m \in \mathcal{F}}} \frac{1}{|\zeta'(\rho_m)|} \ll T^{1+\delta} \log T.$$

The set of excluded zeros whose ordinates do not belong to the family $\mathcal{F}$ conjecturally has arbitrarily small proportion [12, p. 2682]. Thus, it is reasonable to assume that for any $\epsilon > 0$,

$$\sum_{|\gamma_m| \leq T} \frac{1}{|\zeta'(\rho_m)|} \ll T^{1+\epsilon}. \tag{8.3}$$

We note that the above conjecture is weaker than the Gonek-Hejhal conjecture.

To prepare ourselves for stating and proving the main results, we begin with some lemmas.

Lemma 8.1. *For* $\mathrm{Re}(s) > 0$, $\mathrm{Re}(k) > -1$ *and* $y > 0$,

$$\int_0^y \left(1 - \frac{x}{y}\right)^k x^{s-1} \mathrm{d}x = y^s \frac{\Gamma(s)\Gamma(k+1)}{\Gamma(s+k+1)}.$$

Proof. Substitute $x = yt$ and use the evaluation of Euler's beta integral. □

We will also need the following corollary of the well-known Kronecker Theorem [14, Theorem IV, p. 53].

Lemma 8.2. ([40, Theorem B]) *Let* $\{\alpha_n\}_{n=1}^N$ *be* N *real numbers linearly independent over* $\mathbb{Q}$ *and* $\{\beta_n\}_{n=1}^N$ *be* N *arbitrary real numbers. Then for any* $\epsilon > 0$, *there exist an arbitrary large positive real number* t *such that* $||\alpha_n t - \beta_n|| < \epsilon$ *for all* $1 \leq n \leq N$, *where* $||x||$ *denotes distance of* x *from the nearest integer.*

Using Lemma 8.2, we obtain the following result.

Lemma 8.3. *Let $\{C_n\}_{n=1}^{\infty}$ be an infinite sequence of complex numbers such that $\sum_{n=1}^{\infty}|C_n| = S < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers that are linearly independent over $\mathbb{Q}$. Then we have*

$$\limsup_{x\to\infty}\sum_{n=1}^{\infty}\big(C_n\exp(xi\alpha_n)+\bar{C}_n\exp(-xi\alpha_n)\big)=2S,$$
$$\liminf_{x\to\infty}\sum_{n=1}^{\infty}\big(C_n\exp(xi\alpha_n)+\bar{C}_n\exp(-xi\alpha_n)\big)=-2S.$$

Proof. Let $C_n=|C_n|\exp(i\theta_n)$. Thus, $C_n\exp(xi\alpha_n)+\bar{C}_n\exp(-xi\alpha_n)=2|C_n|\cos(\theta_n+x\alpha_n)$. Fix N such that $\sum_{n>N}|C_n|<\epsilon$. Since $\{\alpha_n\}_{n=1}^{N}$ are linearly independent over $\mathbb{Q}$, from Kronecker's theorem (Lemma 8.2) for any $\epsilon>0$, we can find an arbitrarily large number x such that $\left|\left|x\frac{\alpha_n}{2\pi}+\frac{\theta_n}{2\pi}\right|\right|<\frac{\epsilon}{2\pi}$, where $||x||$ denotes distance of x from the nearest integer. Thus $2\pi k-\epsilon<x\alpha_n+\theta_n\leq 2\pi k+\epsilon$ for some integer k. Thus $\cos(x\alpha_n+\theta_n)=1+O(\epsilon^2)$ for all $n\leq N$. So we get

$$\begin{aligned}\sum_{n=1}^{\infty}\big(C_n\exp(xi\alpha_n)+\bar{C}_n\exp(-xi\alpha_n)\big)&=2\sum_{n\leq N}|C_n|(1+O(\epsilon^2))+2\sum_{n>N}|C_n|\cos(x\alpha_n+\theta_n)\\&=2\sum_{n\leq N}|C_n|+O\bigg(\sum_{n\leq N}|C_n|\epsilon^2\bigg)+2\sum_{n>N}|C_n|\cos(x\alpha_n+\theta_n)\\&=2\sum_{n\leq N}|C_n|+O(S\epsilon^2))+O(\epsilon)=2S+O(\epsilon).\end{aligned}$$

Now since $\sum_{n=1}^{\infty}|2C_n\cos(\theta_n+x\alpha_n)|\leq 2S$,

$$\limsup_{x\to\infty}\sum_{n=1}^{\infty}\big(C_n\exp(xi\alpha_n)+\bar{C}_n\exp(-xi\alpha_n)\big)=2S.$$

Also, for any $\epsilon>0$, we can find an arbitrarily large number x such that $\left|\left|x\frac{\alpha_n}{2\pi}+\frac{\theta_n-\pi}{2\pi}\right|\right|<\frac{\epsilon}{2\pi}$. For such x, we have $|\cos(x\alpha_n+\theta_n)+1|<\epsilon$. Using a similar method as shown above, it can be seen that

$$\liminf_{x\to\infty}\sum_{n=1}^{\infty}\big(C_n\exp(xi\alpha_n)+\bar{C}_n\exp(-xi\alpha_n)\big)=-2S.$$

□

Lemma 8.4. *Let $\delta>0$. Assuming RH and (8.3), as $\delta\to 0$, we have*

$$A:=\sum_{\gamma_m}\frac{1}{|\zeta'(\rho_m)||\rho_m|^{1+\delta}}\asymp\frac{1}{\delta}, \tag{8.4}$$

$$B:=\sum_{\gamma_m}\frac{|\zeta(2\rho_m)|}{|\zeta'(\rho_m)||\rho_m|^{1+\delta}}\asymp\frac{1}{\delta}, \tag{8.5}$$

and

$$C:=\sum_{\gamma_m}\frac{|\zeta^4(\rho_m/2)|}{|\zeta'(\rho_m)||\rho_m|^{2+\delta}}\asymp\frac{1}{\delta}. \tag{8.6}$$

Proof. Let

$$S(T):=\sum_{0<\gamma_m\leq T}\frac{1}{|\zeta'(\rho_m)|}.$$

Since $S(t)$ is an increasing function, we have $\mathrm{d}S(t)\geq 0$. Writing the sum A as an integral and then performing integration by parts, we see that

$$A=2\sum_{\gamma_m>0}\frac{1}{|\zeta'(\rho_m)||\rho_m|^{1+\delta}}=2\int_1^{\infty}\frac{\mathrm{d}S(t)}{(\frac{1}{4}+t^2)^{\frac{1+\delta}{2}}}\asymp\int_1^{\infty}\frac{\mathrm{d}S(t)}{t^{1+\delta}}=\frac{S(t)}{t^{1+\delta}}\bigg|_1^{\infty}+(1+\delta)\int_1^{\infty}\frac{S(t)}{t^{2+\delta}}\mathrm{d}t.$$

To obtain the upper bound, we use (8.3) with $\epsilon = \delta/2$ so as to get

$$A \ll \frac{t^{1+\epsilon}}{t^{1+\delta}}\bigg|_1^\infty + (1+\delta)\int_1^\infty \frac{t^{1+\epsilon}}{t^{2+\delta}}\mathrm{d}t = \frac{1}{t^{\delta/2}}\bigg|_1^\infty + (1+\delta)\int_1^\infty \frac{1}{t^{1+\delta/2}}\mathrm{d}t \asymp \frac{1}{\delta}. \tag{8.7}$$

Thus we have $A \ll \delta^{-1}$. To prove the lower bound, using (8.2), we have

$$A \gg \frac{t\log^{1/4} t}{t^{1+\delta}}\bigg|_1^\infty + (1+\delta)\int_1^\infty \frac{t\log^{1/4} t}{t^{2+\delta}}\mathrm{d}t = \frac{\log^{1/4} t}{t^{\delta}}\bigg|_1^\infty + (1+\delta)\int_1^\infty \frac{\log^{1/4} t}{t^{1+\delta}}\mathrm{d}t \asymp \frac{1}{\delta}. \tag{8.8}$$

From (8.7) and (8.8), we conclude that $A \asymp \frac{1}{\delta}$.

Next, we prove that $B \asymp \delta^{-1}$. We know from the results of Titchmarch [60] and Littlewood [39, Theorem 1] that for any given $\epsilon > 0$, $\zeta(1+it) \ll t^{\epsilon}$ and $\zeta(1+it) \gg t^{-\epsilon}$. Taking $\epsilon = \delta/2$ and analyzing the sum B in a similar way as A, we see that

$$B = \sum_{\gamma_m} \frac{|\zeta(2\rho_m)|}{|\zeta'(\rho_m)||\rho_m|^{1+\delta}} \asymp \int_1^\infty |\zeta(1+2it)|\frac{\mathrm{d}S(t)}{t^{1+\delta}} \ll \int_1^\infty \frac{\mathrm{d}S(t)}{t^{1+\delta-\epsilon}} \ll \int_1^\infty \frac{\mathrm{d}S(t)}{t^{1+\delta/2}},$$

thereby leading to $B \ll \delta^{-1}$. We prove the lower bound as follows:

$$B = \sum_{\gamma_m} \frac{|\zeta(2\rho_m)|}{|\zeta'(\rho_m)||\rho_m|^{1+\delta}} \asymp \int_1^\infty |\zeta(1+2it)|\frac{\mathrm{d}S(t)}{t^{1+\delta}} \gg \int_1^\infty \frac{\mathrm{d}S(t)}{t^{1+3\delta/2}} \gg \frac{1}{\delta}.$$

We now prove $C \asymp \delta^{-1}$. Note that

$$C = \sum_{\gamma_m>0} \frac{|\zeta^4(\rho_m/2)|}{|\zeta'(\rho_m)||\rho_m|^{2+\delta}} \asymp \int_1^\infty \left|\zeta^4\left(\frac{1}{4}+\frac{it}{2}\right)\right|\frac{\mathrm{d}S(t)}{t^{2+\delta}}.$$

Using the fact $\zeta(s) \asymp t^{\frac{1}{2}-\sigma}\zeta(1-s)$, we have $\zeta(1/4+it) \asymp t^{1/4}\zeta(3/4-it)$. Using RH, Titchmarsh[61, p. 337, Equations (14.2.5), (14.2.6)] proved that $\zeta(3/4-it) \ll t^{\epsilon}$ and $\zeta(3/4-it) \gg t^{-\epsilon}$. Thus $\zeta(1/4+it) \ll t^{1/4+\epsilon}$ and $\zeta(1/4+it) \gg t^{1/4-\epsilon}$. Thus we have $\zeta^4(1/4+it) \ll t^{1+4\epsilon}$ and $\zeta^4(1/4+it) \gg t^{1-4\epsilon}$. We choose $\epsilon = \delta/8$. Thus we have,

$$C \asymp \int_1^\infty \left|\zeta^4\left(\frac{1}{4}+\frac{it}{2}\right)\right|\frac{\mathrm{d}S(t)}{t^{2+\delta}} \ll \int_1^\infty \frac{\mathrm{d}S(t)}{t^{1+\delta-4\epsilon}} = \int_1^\infty \frac{\mathrm{d}S(t)}{t^{1+\delta/2}} \ll \frac{1}{\delta}$$

and

$$C \asymp \int_1^\infty \left|\zeta^4\left(\frac{1}{4}+\frac{it}{2}\right)\right|\frac{\mathrm{d}S(t)}{t^{2+\delta}} \gg \int_1^\infty \frac{\mathrm{d}S(t)}{t^{1+3\delta/2}} \gg \frac{1}{\delta}.$$

Thus $C \asymp \frac{1}{\delta}$. □

Theorem 8.5. *Let $\delta > 0$. Assume RH, simplicity of the non-trivial zeros of $\zeta(s)$, Equation (8.3), and the Linear Independence conjecture. Then there exists an absolute positive constant c such that for all sufficiently small δ,*

$$\limsup_{y\to\infty} \sum_{n\le y} \frac{\mu(n)(1-n/y)^{\delta}}{\sqrt{y}} \ge \frac{c}{\delta} \quad \text{and} \quad \liminf_{y\to\infty} \sum_{n\le y} \frac{\mu(n)(1-n/y)^{\delta}}{\sqrt{y}} \le -\frac{c}{\delta}. \tag{8.9}$$

Proof. In 7.3, we take

$$\phi(x) = \begin{cases} \left(1-\frac{x}{y}\right)^{\delta} & \text{if } x \le y, \\ 0 & \text{otherwise.} \end{cases} \tag{8.10}$$

For a fixed $a > 0$ as $|s| > 1$, we have

$$\frac{\Gamma(s)}{\Gamma(s+a)} = \frac{1}{s^a} + O\left(\frac{1}{s^{a+1}}\right). \tag{8.11}$$

From Lemma 8.1 and (8.11),

$$\frac{1}{\zeta'(\rho_m)}\int_0^y\left(1-\frac{x}{y}\right)^{\delta}x^{\rho_m-1}\mathrm{d}x=\frac{y^{\rho_m}}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)\Gamma(\rho_m)}{\Gamma(1+\rho_m+\delta)}\asymp\frac{y^{\rho_m}}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}}+O\left(\frac{\sqrt{y}}{\zeta'(\rho_m)\rho_m^2}\right).$$

Since, using the ideas in the proof of Lemma 8.3 and under the same hypotheses, it is evident that the series $\sum\limits_{\rho_m}\frac{1}{|\zeta'(\rho_m)\rho_m^2|}$ converges, we get

$$\sum_{\gamma_m}\frac{1}{\zeta'(\rho_m)}\int_0^y\left(1-\frac{x}{y}\right)^{\delta}x^{\rho_m-1}\mathrm{d}x=\sqrt{y}\sum_{\gamma_m}\frac{y^{i\gamma_m}}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}}+O(\sqrt{y}).$$

Now we apply Lemma 8.3 with $x=\log y$, $C_m=\rho_m^{-\delta-1}/\zeta'(\rho_m)$, assume the Linear Independence conjecture so that we can let $\alpha_m=\gamma_m, m>0$, and use (8.4) to obtain

$$\limsup_{y\to\infty}\sum_{\gamma_m}\frac{y^{i\gamma_m}}{\rho_m^{1+\delta}\zeta'(\rho_m)}\geq\frac{c'}{\delta}\quad\text{and}\quad\liminf_{y\to\infty}\sum_{\gamma_m}\frac{y^{i\gamma_m}}{\rho_m^{1+\delta}\zeta'(\rho_m)}\leq-\frac{c'}{\delta}\tag{8.12}$$

for some absolute constant $c'>0$. The Mellin transform of $\phi(x)$ is $\Phi(s)=y^s\frac{\Gamma(1+\delta)\Gamma(s)}{\Gamma(1+\delta+s)}$ and

$$\int_{(-\tau)}\frac{\Phi(s)}{\zeta(s)}\mathrm{d}s\ll\int_{(-\tau)}\left|\frac{\Phi(s)}{\zeta(s)}\right||\mathrm{d}s|\ll y^{-\tau}\int_{(-\tau)}\left|\frac{\Gamma(1+\delta)\Gamma(s)}{\Gamma(1+\delta+s)\zeta(s)}\right||\mathrm{d}s|=O(y^{-\tau}).$$

Moreover, the residue of $\Phi(s)$ at $s=0$ is 1. Therefore, equation (7.3) becomes

$$\sum_{n\leq y}\mu(n)(1-n/y)^{\delta}=\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\sqrt{y}\frac{y^{i\gamma_m}}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}}+O(\sqrt{y})-2+O(y^{-\tau}).$$

Dividing both sides by $\sqrt{y}$, we get

$$\frac{1}{\sqrt{y}}\sum_{n\leq y}\mu(n)(1-n/y)^{\delta}=\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{y^{i\gamma_m}}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}}+O(1),$$

which, along with (8.12), implies (8.9). □

Theorem 8.6. *Let $\delta>0$. Assume RH, simplicity of the non-trivial zeros of $\zeta(s)$, Equation (8.3), and the Linear Independence conjecture. Then there exists an absolute positive constant c such that for all sufficiently small δ,*

$$\limsup_{y\to\infty}\sum_{n\leq y}\frac{\lambda(n)(1-n/y)^{\delta}}{\sqrt{y}}\geq\frac{c}{\delta}\quad\text{and}\quad\liminf_{y\to\infty}\sum_{n\leq y}\frac{\lambda(n)(1-n/y)^{\delta}}{\sqrt{y}}\leq-\frac{c}{\delta}.\tag{8.13}$$

Proof. From (4.11), (4.20) and (4.22), for a small positive τ,

$$\sum_{n=1}^{\infty}\lambda(n)\phi(n)=\frac{1}{2\pi i}\int_{(-\tau)}\frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s+\mathrm{Res}_{s=0}\frac{\zeta(2s)}{\zeta(s)}\Phi(s)+\frac{\Phi\left(\frac{1}{2}\right)}{2\zeta(\frac{1}{2})}+\lim_{T_n\to\infty}\sum_{|\gamma_m|<T_n}\frac{\zeta(2\rho_m)\Phi(\rho_m)}{\zeta'(\rho_m)}.$$

Let ϕ be defined in (8.10). From Lemma 8.1 and (8.11),

$$\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\int_0^y\left(1-\frac{x}{y}\right)^{\delta}x^{\rho_m-1}\mathrm{d}x=y^{\rho_m}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)\Gamma(\rho_m)}{\Gamma(1+\rho_m+\delta)}\asymp y^{\rho_m}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)}\frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}}+O\left(\sqrt{y}\frac{\zeta(2\rho_m)}{\zeta'(\rho_m)\rho_m^2}\right).$$

Now use Lemma 8.3 with $C_m=\rho_m^{-\delta-1}\zeta(2\rho_m)/\zeta'(\rho_m)$ and $\alpha_m=\gamma_m$ for $m>0$ and $x=\log y$ and use (8.5) to obtain

$$\limsup_{y\to\infty}\sum_{m=1}^{\infty}y^{i\gamma_m}\frac{\zeta(2\rho_m)}{\rho_m^{1+\delta}\zeta'(\rho_m)}\geq\frac{c'}{\delta}\quad\text{and}\quad\liminf_{y\to\infty}\sum_{m=1}^{\infty}y^{i\gamma_m}\frac{\zeta(2\rho_m)}{\rho_m^{1+\delta}\zeta'(\rho_m)}\leq-\frac{c'}{\delta}\tag{8.14}$$

for some absolute constant $c' > 0$. Moreover,

$$\int_{(-\tau)} \frac{\zeta(2s)}{\zeta(s)}\Phi(s)\mathrm{d}s \ll \int_{(-\tau)} \left|\frac{\zeta(2s)}{\zeta(s)}\right| |\Phi(s)||\mathrm{d}s| \ll y^{-\tau} \int_{(-\tau)} \left|\frac{\Gamma(1+\delta)\Gamma(s)\zeta(2s)}{\Gamma(1+k+s)\zeta(s)}\right| |\mathrm{d}s| = O(y^{-\tau}).$$

Also $\mathrm{Res}_{s=0} \frac{\zeta(2s)}{\zeta(s)}\Phi(s) = 1$. So

$$\sum_{n\le y} \lambda(n)\left(1-\frac{n}{y}\right)^{\delta} = \frac{y^{1/2}}{2\zeta(\frac{1}{2})}\frac{\Gamma(1/2)\Gamma(1+\delta)}{\Gamma(1+\delta+1/2)} + \lim_{T_n\to\infty} \sqrt{y} \sum_{|\gamma_m|<T_n} y^{i\gamma_m} \frac{\zeta(2\rho_m)}{\zeta'(\rho_m)} \frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}} + O\left(\sqrt{y}\right)$$
$$+ O(y^{-\tau}) + 1.$$

Dividing both sides by $\sqrt{y}$, we get

$$\sum_{n\le y} \frac{\lambda(n)(1-n/y)^{\delta}}{\sqrt{y}} = \lim_{T_n\to\infty} \sum_{|\gamma_m|<T_n} \frac{y^{i\gamma_m}}{\zeta'(\rho_m)} \frac{\Gamma(1+\delta)}{\rho_m^{1+\delta}} + O(1),$$

which, along with (8.14), implies (8.13). □

Theorem 8.7. *Let $\delta > 0$. Assume RH, simplicity of the non-trivial zeros of $\zeta(s)$, Equation (8.3), and the Linear Independence conjecture. Then there exists an absolute positive constant c such that for all sufficiently small δ,*

$$\limsup_{y\to\infty} \frac{\sum_{n\le y} d^2(n)(1-n/y)^{1+\delta} - g(y)}{y^{1/4}} \ge \frac{c}{\delta} \quad \textit{and} \quad \liminf_{y\to\infty} \frac{\sum_{n\le y} d^2(n)(1-n/y)^{1+\delta} - g(y)}{y^{1/4}} \le -\frac{c}{\delta}, \tag{8.15}$$

where

$$g(y) := \frac{y}{2+\delta} \sum_{j=0}^{3} B_{\delta,j}(y), \tag{8.16}$$

$$B_{\delta,j}(y) := A_j \int_0^y \left(1-\frac{x}{y}\right)^{1+\delta} \log^j(x)\mathrm{d}x \qquad (0\le j\le 3), \tag{8.17}$$

with $A_j, 1 \le j \le 3$, defined in (2.15).

Proof. In Theorem 2.8, we considered $\Phi(s)$ to be holomorphic in $-1 < \mathrm{Re}(s) < 2$ since we were explicitly evaluating the integral $\int_{(-\tau)} \frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s$ to be the sum on the right-hand side of (2.14), and had we allowed $\Phi(s)$ to have a pole at $s = 0$, computing this would have been unwieldy . However, in what follows, we allow $\Phi(s)$ to have a simple pole at $s = 0$ because we are only estimating the integral over $\mathrm{Re}(s) = -\tau$. We thus have to consider the contribution of the residue at this additional pole, and because of this, and (5.10), (5.11) and (5.12), for a small positive τ,

$$\sum_{n=1}^{\infty} d^2(n)\phi(n) = \frac{1}{2\pi i}\int_{(-\tau)} \frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)\mathrm{d}s + \int_0^{\infty} (A_0 + A_1\log(x) + A_2\log^2(x) + A_3\log^3(x))\phi(x)\mathrm{d}x$$
$$+ \mathrm{Res}_{s=0}\frac{\zeta^4(s)}{\zeta(2s)}\Phi(s) + \lim_{T_n\to\infty} \sum_{|\gamma_m|<T_n} \frac{\zeta^4(\rho_m)\Phi(\rho_m)}{\zeta(2\rho_m)},$$

where A_0, A_1, A_2 and A_3 are defined in (2.15). In the above identity, let

$$\phi(x) = \begin{cases} \left(1-\frac{x}{y}\right)^{1+\delta} & \text{if } x \le y, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned}\frac{\zeta^4(\frac{\rho_m}{2})}{\zeta'(\rho_m)}\int_0^y\left(1-\frac{x}{y}\right)^{1+\delta}x^{\frac{\rho_m}{2}-1}\mathrm{d}x&=\frac{\zeta^4(\frac{\rho_m}{2})}{\zeta'(\rho_m)}y^{\rho_m/2}\frac{\Gamma(2+\delta)\Gamma(\rho_m/2)}{\Gamma(2+\delta+\rho_m/2)}\\&=y^{1/4}\frac{\zeta^4(\frac{\rho_m}{2})\Gamma(2+\delta)}{\zeta'(\rho_m)(\rho_m/2)^{2+\delta}}y^{i\gamma_m/2}+O\left(y^{1/4}\frac{\zeta^4(\frac{\rho_m}{2})}{\zeta'(\rho_m)\rho_m^3}\right).\end{aligned}$$

Also,

$$\mathrm{Res}_{s=0}\,\frac{\zeta^4(s)}{\zeta(2s)}\Phi(s)=\mathrm{Res}_{s=0}\,\frac{\zeta^4(s)}{\zeta(2s)}y^s\frac{\Gamma(\delta+2)\Gamma(s)}{\Gamma(s+\delta+2)}=-\frac{1}{8}.$$

So we get

$$\sum_{n\leq y}d^2(n)\left(1-\frac{n}{y}\right)^{1+\delta}=g(y)-\frac{1}{8}+\lim_{T_n\to\infty}\sum_{|\gamma_m|\leq T_n}y^{1/4}\frac{2^{2+\delta}\zeta^4(\frac{\rho_m}{2})}{\zeta'(\rho_m)\rho_m^{2+\delta}}y^{i\gamma_m/2}+O\left(y^{1/4}\frac{\zeta^4(\frac{\rho_m}{2})}{\zeta'(\rho_m)\rho_m^3}\right),$$

where $g(y)$ is defined in (8.16). Dividing both sides by $y^{1/4}$, we get

$$\frac{\sum_{n\leq y}d^2(n)(1-n/y)^{1+\delta}-g(y)}{y^{1/4}}=\lim_{T_n\to\infty}\sum_{|\gamma_m|\leq T_n}\frac{2^{2+\delta}\zeta^4(\frac{\rho_m}{2})}{\zeta'(\rho_m)\rho_m^{2+\delta}}y^{i\gamma_m/2}+O(1).$$

Using (8.6) and ideas similar to those used to prove Theorems 8.5 and 8.6, we are led to (8.15). □

Remark 12. *While we do not explicitly write down $B_{\delta,j}(x)$, $1\leq j\leq 3$, we note that $g(y)/y$ is a cubic polynomial in* $\log(y)$.

Remark 13. *The assumption of* (8.3) *in Theorems* 8.5, 8.6 *and* 8.7 *can be replaced by the weaker assumption that the series A, B, and C in Lemma* 8.4 *converge.*

Results analogous to those derived in Theorems 8.5-8.7 can be obtained by choosing $\phi_y(x)=e^{-xy}$, however, the techniques involved in proving them are similar, and hence to avoid repetition, we have chosen to forego the calculations.

9. Concluding remarks

Voronoï summation formulas have been historically used to improve the error term associated to the summatory function of the arithmetic function involved. Several articles have been written on refining the error term of $\sum_{n\leq x}d^2(n)$, in particular. Let $E(x)$ denote the error term. The current best upper bound on $E(x)$ is by Jia and Sankaranarayanan [35] who proved that $E(x)=O\left(x^{1/2}\log^5(x)\right)$ whereas Chandrasekharan and Narasimhan [15] have shown that $E(x)=\Omega_{\pm}(x^{1/4})$. See [35], [57, p. 438] for the history on this topic. It would be nice to see if the Voronoï summation formula that we have obtained for $d^2(n)$ (Theorem 2.8) could be used for improving $E(x)$. Although the hypotheses of Theorem 2.8 does not allow us to take $\phi(x)$ to be the characteristic function of the interval $[1,x]$, where $x>1$, heuristically, if we choose ϕ to be so, then the main term of our result turns out to be exactly the one given by Ramanujan [54], namely[10], $A_0x+A_1x\log(x)+A_2x\log^2(x)+A_3x\log^3(x)$. Thus, it is certainly of merit to expand the class of admissible functions ϕ for which Theorem 2.8 is valid and thereby obtain better estimates on the error term.

Koshliakov [37] devised an ingenious method to derive the Voronoï summation for $\sum_{\alpha<n<\beta}d(n)\phi(n)$, where $0<\alpha<\beta,\alpha,\beta\notin\mathbb{Z}$, and f denotes a functional analytic inside a closed contour strictly containing $[\alpha,\beta]$. This technique requires nothing more than the functional equation of $\zeta(s)$ and the Cauchy residue theorem. It was adapted in [6, Theorem 6.1] and then in [23, Theorem 2.2] to handle the corresponding formulas for $\sigma_s(n)$ and $\sigma_s^{(k)}(n)$ respectively. In his method, Koshliakov starts from the identity (1.16) (which, as mentioned in the introduction, results from (1.15) by replacing x by ix and $-ix$ and adding the resulting two identities), replaces x by iz and $-iz$ in (1.16), adds the corresponding sides of the two

[10]Our A_0,A_1,A_2 and A_3 are respectively D,C,B and A of Ramanujan.

resulting identities, applies the functional equation of $\zeta(s)$, and then analyzes a certain contour integral to arrive at the Voronoï summation formula.

Unfortunately, we were unable to adapt this technique for the arithmetic functions we have considered in our paper. For example, consider the case of $d^2(n)$. We *do* have the analogue of (1.15), namely, (2.26). However, if we replace x in there by ix and $-ix$, and add the resulting two identities to obtain a corresponding analogue of (1.16), the problem is the resulting series of the non-trivial zeros of $\zeta(s)$ diverges. However, if one is able to modify this step and thereby circumvent this obstacle, one may be able to obtain the Voronoï summation not only for $\sum_{\alpha<n<\beta} d^2(n)\phi(n)$, but also for $\sum_{\alpha<n<\beta} \sigma_a(n)\sigma_b(n)\phi(n)$. Then one may as well try out the same in the setting of $\lambda(n)$ for which, again, the analogue of (1.15) exists, namely, (2.11). This definitely seems to be a worthwhile problem to try.

It would also be important to obtain analogues of our summation formulas and identities of Ramanujan-Guinand type and Cohen type for different L-functions. For example, Berndt, Kim and Zaharescu [8] derived the Ramanujan-Guinand type formula for the twisted divisor function $d_\chi(n) := \sum_{d|n} \chi(d)$, where χ is an even primitive Dirichlet character, and used to show that when χ is real, $L(1, \chi) > 0$.

In Section 8, we obtained results on oscillations of the Riesz sums associated with $\lambda(n), \mu(n)$ and $d^2(n)$. As remarked at the end of that section, one may derive similar results where the test function is e^{-xy}. The key thing which makes such results work is the presence of the series over the non-trivial zeros of $\zeta(s)$. In light of this, it may be interesting to find other functions than the ones we have studied which also exhibit oscillations caused due to the series over the non-trivial zeros.

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