# THE MULTIFACETED ROGERS-RAMANUJAN FUNCTIONS

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Abstract. This expository article is a survey of some of the developments in partition theory arising from Rogers-Ramanujan identities and in the topic of modular relations satisfied by the Rogers-Ramanujan functions.

## 1. Introduction

Every college student has, at some point of time, encountered the number

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}. \tag{1.1}$$

The standard way with which a student proceeds, of course disregarding the convergence aspects, is to observe that if $x$ denotes the above number, then $1/x = 1 + x$. This equation has two roots $\frac{1}{2}(-1 \pm \sqrt{5})$. Since $x$ is positive, one concludes that it is equal to $\frac{1}{2}(-1 + \sqrt{5})$.

One of the triumphs of modern mathematics is the advancement made in the theory of functions. For example, what if we consider, instead of the number in (1.1), the function

$$F(q) := \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\dots}}}}? \tag{1.2}$$

One represents the above continued fraction in the following standard compact notation:

$$F(q) = \frac{1}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\cdots. \tag{1.3}$$

It is known that $F(q)$ converges for $|q| < 1$. Its behavior on the unit circle, however, is mysterious and still not understood completely. There are certain roots of unity where we know it converges (including, of course, $q = 1$) and certain others where it diverges; see [20, Theorem 7.2.1].

Consider a slightly modified continued fraction, that is,

$$R(q) := q^{1/5}F(q). \tag{1.4}$$

This is the most celebrated continued fraction in Mathematics known as the *Rogers-Ramanujan continued fraction*. It was first studied by the English mathematician L. J. Rogers [39], and was rediscovered by the famous Indian mathematician S. Ramanujan before going to England. The latter obtained a plethora of results involving $R(q)$ in his Notebooks and the Lost Notebook [38].

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The theory of $R(q)$ entails the study of two functions $G(q)$ and $H(q)$ called the *Rogers-Ramanujan functions.* These functions are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \text{ and } H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}, \tag{1.5}$$

where, here, and throughout the paper, we use the standard notation adopted in $q$-series:

$$\begin{aligned} (A)_0 &:= (A;q)_0 = 1, \\ (A)_n &:= (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \qquad n \geq 1, \\ (A)_\infty &:= (A;q)_\infty = \lim_{n\to\infty}(A;q)_n, \qquad |q|<1. \end{aligned}$$

The functions $G(q)$ and $H(q)$ satisfy what are known as *Rogers-Ramanujan identities.* Many mathematicians consider them to be *'the most beautiful pair of formulas in all of mathematics'* [47]. These identities are given, for $|q| < 1$, by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.7}$$

Rogers [39] derived these identities in 1894. Ramanujan rediscovered them in 1913 before going to England. An interesting account on Ramanujan's discovery of these identities in Rogers' work is given in [31].

Issai J. Schur, a German mathematician, who was cut off from England during the World War I, also rediscovered the identities, and gave two proofs [42], one of which was combinatorial in nature. Before we discuss this combinatorial interpretation[1] of the Rogers-Ramanujan identities (1.6) and (1.7) given by Schur, we need to discuss a fundamental construct in additive number theory: *partitions.*

## 2. Basics of partition theory

A partition of a non-negative integer $n$ is a non-increasing sequence of positive integers which sum to $n$. For example, $3+2+1+1$ is a partition of 7.

The number of partitions of $n$ is denoted by $p(n)$, *the partition function.* Thus, $p(4) = 5$ since there are five partitions of 4, namely, 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. By convention, we take $p(0) = 1$.

While studying an arithmetic sequence $\{a(n)\}_{n=0}^{\infty}$, it is important to know its generating function. Different kinds of generating functions are suited for various purposes. The most common one is $\sum_{n=0}^{\infty} a(n)q^n$, where $q$ is some real/complex variable. For example, the generating function for the sequence of natural numbers is $1\cdot q + 2\cdot q^2 + 3\cdot q^3 + \cdots = q/(1-q)^2$, provided $|q| < 1$.

Euler was the first to obtain a representation for the generating function of the partition function $p(n)$, namely, he showed that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\cdots = \frac{1}{(q;q)_\infty}. \tag{2.1}$$

[1]This combinatorial interpretation was first found by MacMahon [35, Chapter 3]. However, MacMahon was unable to prove the identities themselves.

The infinite product given above converges for $|q| < 1$. Henceforth, unless specified otherwise, we will always assume $|q| < 1$.

Euler's result is one of the first theorems proved in a standard course in the theory of partitions or in a book devoted to the subject, see [12] or [17], for example. The proof is not difficult and one only requires to know that the $m$ in

$$\frac{1}{1-q^m} = \sum_{j=0}^{\infty} q^{jm}, \tag{2.2}$$

denotes a part of the concerned partition and $j$, the number of times $m$ appears as a part. When $j = 0$, it, of course, means that $m$ does not appear as a part in the partition.

Why are we interested in a generating function of a sequence? The first and the foremost reason is, it provides us an analytic object encapsulating the information of *every* element of the sequence, and which is also well-suited to algebraic and analytic manipulations. As rightly said by the famous combinatorialist Herbert Wilf [23, p. 145], '*a generating function of a sequence is a clothesline on which you hang all elements of the sequence*'.

## 3. Partition identities

In this section, we look at identities of the form $p_1(n) = p_2(n)$, where $p_1$ and $p_2$ are two restricted partition functions with different restrictions on their parts.

3.1. **Euler's partition theorem.** The generating function in (2.1) is extremely useful in finding important results in the subject. Consider, for example, the number of partitions of 6 into distinct parts. There are four such partitions, namely, $6, 5+1, 4+2$ and $3+2+1$. Now consider the number of partitions of 6 into odd parts. These are also four in number: $5+1, 3+3, 3+1+1+1$, and $1+1+1+1+1+1$. Is this a mere coincidence? One may check for a few more numbers that these two sets of partitions are equinumerous. How do we show this for *every* positive integer $n$?

This is where the generating functions can do magic. Let $p_d(n)$ denote the number of partitions of $n$ into distinct parts. Since any part of a partition into distinct parts either appears only once or does not appear at all, to obtain $\sum_{n=0}^{\infty} p_d(n)q^n$, one needs to truncate the sum in (2.2) to $1 + q^m$. Doing this for each of the factors in the middle expression in (2.1), we find that

$$\sum_{n=0}^{\infty} p_d(n)q^n = (1+q)(1+q^2)(1+q^3)\cdots . \tag{3.1}$$

whereas, if $p_o(n)$ denotes the number of partitions of $n$ into odd parts, we must have

$$\sum_{n=0}^{\infty} p_o(n)q^n = \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots}. \tag{3.2}$$

Using the elementary identity $(a-b)(a+b) = a^2 - b^2$, one can formally[2] see that the right-hand sides of (3.1) and (3.2) are equal, which implies that $p_d(n) = p_o(n)$. This means the number of partitions of a positive integer into distinct parts is *always* equal to the number of partitions of that integer into odd parts.

Let us rephrase Euler's partition theorem as follows:

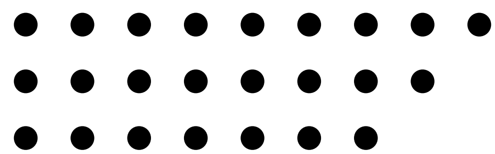
*The number of partitions of a positive integer into parts which differ by at least* 1 *is equal to the number of partitions of that integer into parts which are congruent to* $\pm 1$ *modulo* 4.

[2]This argument can be made rigorous. See [12, pp. 4-5].

3.2. **MacMahon's and Schur's combinatorial interpretations of the Rogers-Ramanujan identities.** In view of the paraphrasing of Euler's partition theorem given at the end of the above sub-section, one might wonder if there more results of this type. For example, *what can be said about the number of partitions of a positive integer whose parts differ by at least* 2*?*

We now show that they are enumerated by the power series coefficients of the left-hand side of the first Rogers-Ramanujan identity, that is, by $\sum_{n=0}^{\infty} q^{n^2}/(q;q)_n$.

To prove this, however, we first need a way to diagrammatically represent a partition. This is done using the *Ferrers diagram* which uses dots to represent each part of a partition. For example, the Ferrers diagram of the partition $9+8+7$ of 24 is



Consider the black dots as lattice points in the $xy$-plane with the top leftmost dot being the origin. The line $y=-x$ restricted to the fourth quadrant is known as the *main diagonal*. If one reflects a partition in its main diagonal, what one gets is the *conjugate* of that partition. Thus, the conjugate of $9+8+7$ is $3+3+3+3+3+3+3+2+1$. It readily follows from conjugation that the number of partitions of an integer into parts whose size is less than or equal to $n$ equals the number of partitions of that integer into less than or equal to $n$ parts.

Now consider the summand $q^{n^2}/(q;q)_n$. The numerator $q^{n^2}$ can be written as

$$q^{n^2} = q^1 \cdot q^3 \cdot q^5 \cdots q^{2n-1},$$

Moreover,

$$\frac{1}{(q;q)_n} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

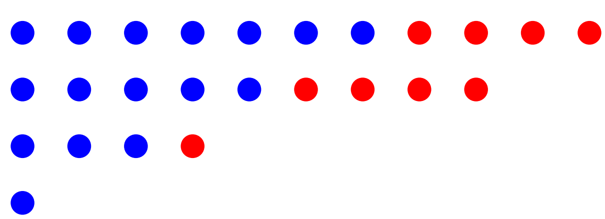
enumerates the number of partitions of an integer into less than or equal to $n$ parts.

We now construct a partition whose Ferrers diagram is constructed as follows:

1. The parts $2n-1, 2n-3, \cdots, 5, 3, 1$ (denoted in blue color in the diagram below) are placed below one another in the same order. These constitute $n$ rows of the Ferrers diagram.

2. Concatenate the largest part of the partition generated by $1/(q;q)_n$ (which is taken to be the sample partition $4+4+1$ in red color in the diagram below) with the $2n-1$ blue dots, the next largest with $2n-3$ blue dots, and so on. If there are fewer than $n$ parts coming from $1/(q;q)_n$, the corresponding last few parts formed by blue color remain unchanged.

The figure below shows this construction for $n=4$.



Since the blue sub-parts of the partition formed by concatenation differed by 2 to begin with, the parts of the complete partition differ by at least 2. To get all such partitions, we must sum $q^{n^2}/(q;q)_n$ over $n$ from $n=0$ to $\infty$ which proves the claim.

Now, clearly, $\dfrac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$ generates partitions having parts congruent to either 1 or 4 modulo 5. Therefore, the combinatorial equivalent of the first Rogers-Ramanujan identity (1.6) is

*The number of partitions of a positive integer into parts differing by at least* 2 *equals the number of partitions of that integer into parts which are congruent to* $\pm 1$ *modulo* 5*.*

We have thus found the next Euler-type partition theorem! Similarly, the second Rogers-Ramanujan identity (1.7) is equivalent to the statement that *the number of partitions of a positive integer into parts differing by at least* 2 *and not having* 1 *as a part equals the number of partitions of that integer into parts which are congruent to* 2 *or* 3 *modulo* 5. This is owing to the fact that $n^2 + n = 2 + 4 + 6 + \cdots + 2n$.

It is to be noted, however, that each of the available proofs of (1.6) and (1.7), or their aforementioned combinatorial equivalents, is, in some or the other way, quite deep! Hardy [31] noted, "*...None of the proofs of these identities can be called both 'simple' and 'straightforward', since the simplest are essentially verifications; and no doubt it would be unreasonable to expect a really easy proof*". Hardy's statement remains true to this day, for, each proof (barring those which are mere verifications) involves an ingenious idea.

Andrews [14] surveyed all known proofs of the Rogers-Ramanujan identities until 1989. New proofs continue to emerge to this day, the most recent one being that given by Rosengren [41]. That being said, there is no "simple" bijective proof of fact that the partitions enumerated by the sum-sides of (1.6) and (1.7) are respectively equal in number to those enumerated by the product-sides! A proof along these lines by Garcia and Milne [28] falls short of being called a 'direct bijection' since it involves intermediate transformations, and is 51 pages long! On the other hand, the "short" bijective proof due to Bressoud and Zeilberger [25] is quite difficult and not really short (see the MathSciNet review of [25] and also [6, p. 3]).

Besides the Theory of Partitions and Modular Forms (more generally, Number Theory), Rogers-Ramanujan identities have played a significant role in several diverse areas of Mathematics and Science such as Commutative Algebra, Knot Theory, Statistical Mechanics, Representation Theory of Affine Lie Algebras and Algebraic Geometry, to name a few. See [1, 18, 19, 33, 34, 26]. An interested reader is encouraged read the excellent book by Sills [44], entirely devoted to the Rogers-Ramanujan identities!

### 3.3. **Schur's partition theorem, Alder's conjecture, and Andrews-Gordon identities.**

The topic of finding partition identities similar to the combinatorial interpretations of the Rogers-Ramanujan identities has blossomed ever since the identities in (1.6) and (1.7) were obtained. While a complete survey of these developments is beyond the scope of this review, we indicate only a few of them to give an idea of their vast expanse.

In the previous subsection, we saw that the combinatorial interpretation of the first Rogers-Ramanujan identity is, in some sense, the next level result of Euler's partition theorem.

What next? In 1926, Schur [43] made further progress in this direction by deriving the following beautiful result.

*The number of partitions of an integer into parts that differ by at least* 3*, and with no consecutive multiples of* 3 *as parts, equals the number of partitions of the integer into parts which are congruent to* $\pm 1$ *modulo* 6*.*

One immediately observes the extra condition needed to make the two sets of partitions equinumerous, namely, that the partitions on which the difference conditions are imposed have to have a difference of at least six in between any two of its parts which are multiples of 3. Such an additional condition was not needed for example in Euler's partition theorem or in the combinatorial interpretation of (1.6).

However, Lehmer [32] proved a general result whose special case is that if $q_d(n)$ denotes the number of partitions of a positive integer into parts differing by at least $d$ and $Q_d(n)$ denotes the number of parts congruent to $\pm 1$ modulo $(d+3)$, then, for $d \geq 3$, then $q_d(n) \neq Q_d(n)$. However, based on the fact that $q_3(n) - Q_3(n) \geq 0$, as can be checked from the aforementioned result of Schur, Alder [3], [4] conjectured that $q_d(n) - Q_d(n) \geq 0$ for all $d, n \in \mathbb{N}$. This conjecture was proved in complete generality only fifty years later through the combined efforts of Andrews [10] (for $d = 2^r - 1, r \geq 4$), of Yee [46] (for $d = 7, d \geq 32$), and of Alfes, Jameson and Lemke Oliver [5] (for all remaining values of $d$).

On the other hand, Alder [3, Theorem 3] showed that to have partition theorems similar to that Schur's 1926 theorem or the combinatorial interpretation of (1.6) for the difference $d$ between the parts greater than 3, one must have more complicated conditions. In this direction, Andrews [8], [9] obtained two infinite families of results for $d \geq 3$ encompassing Schur's 1926 result. Both the families intersect only for $d = 3$. Since then, there have been many exciting developments for which the reader is referred to a nice survey by Alladi [6].

Moreover, Gordon [30] obtain a different generalization of (1.6) and (1.7) wherein the modulus associated to certain congruence conditions is any odd number. His theorem reads [30, Theorem 1]

*For* $d \geq 2$*, the number of partitions of* $N$ *into parts not congruent to* $0, \pm t \pmod{2d+1}$*, where* $1 \leq t \leq d$*, is equal to the number of partitions of the form* $N = N_1 + \cdots + N_k$*, where* $N_i \geq N_{i+1}, N_i \geq N_{i+d-1} + 2$*, and* $N_{k-t+1} \geq 2$*.*

Andrews [11, Theorem 1] obtained the analytic counterpart of the above result. It is easy to conceive the generating function of the partitions not satisfying the congruence conditions. Indeed, it is $\prod_{\substack{n=1 \\ n \not\equiv 0, \pm t \pmod{2d+1}}}^{\infty} \frac{1}{1-q^n}$ for $1 \leq t \leq d$. However, the generating function of the partitions from the other set is a multi-dimensional sum. Andrews [7, Theorem 2] obtained an analogue of Gordon's result for the moduli of the form $4d+2$. Finally, Bressoud [24] obtained a result for *all* moduli thus encompassing the results of Andrews and Gordon.

## 4. Modular identities associated with the Rogers-Ramanujan functions

So far we have seen the combinatorics of partition functions arising from the Rogers-Ramanujan identities and their analogues. However, the Rogers-Ramanujan functions $G(q)$ and $H(q)$ defined in (1.5) are intimately connected to modular forms as well. Modular forms is an all-pervasive branch of number theory. A quote attributed to Martin Eichler reads, *'There are five fundamental operations in mathematics: addition, subtraction, multiplication, division, and modular forms.'*. This underscores the importance of modular forms in mathematics as well as other sciences. But what is a modular form?

A modular form $f$ of weight $k$, where $k \in 2\mathbb{N}$, is a function defined [27, p. 4, Definition 1.1.2] on the upper half-plane $\mathbb{H} := \{x+iy \in \mathbb{C} : y > 0\}$ satisfying following three properties:

(i) $f$ is holomorphic on the upper half-plane;
(ii) For any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ and any $z \in \mathbb{H}$, we have $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$;
(iii) $f$ is holomorphic at $\infty$.

Observe that in (1.4) we have a $q^{1/5}$ in front of the continued fraction $F(q)$. The reader might have wondered whether this power of $q$ has been put artificially. It turns out that for $q = e^{2\pi i z}$, $z \in \mathbb{H}$, the functions $\tilde{G}(q) := q^{-1/60}G(q)$ and $\tilde{H}(q) := q^{11/60}H(q)$ are modular forms with respect to $z$, and also that $R(q) = \tilde{H}(q)/\tilde{G}(q) = q^{1/5}H(q)/G(q)$. Thus, the $F(q)$ defined in (1.3) is simply the quotient of $H(q)/G(q)$.

The Rogers-Ramanujan functions $G(q)$ and $H(q)$ satisfy scores of modular relations meaning there is a relation connecting them with $G(q^n)$ and $H(q^n)$ for some natural number $n > 1$. Ramanujan initiated the topic of finding modular relations for the Rogers-Ramanujan functions by obtaining the result [36] (see also [37, p. 231])

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1. \tag{4.1}$$

One of the manuscripts of Ramanujan transcribed by Watson (see [22] for details), contains 40 identities involving $G$ and $H$ which resemble (4.1)! Rogers [40] not only proved (4.1) but also nine other identities which were communicated to him by Ramanujan. The proofs of most of the forty identities were given by various mathematicians such as Darling, Watson, Bressoud, Biagioli, Yesilyurt in a series of papers culminating into the monograph [21] by Berndt, Choi, Choi, Hahn, Yeap, Yee, Yesilyurt, and Yi. Regarding these forty identities, Watson [45] says,

*. . . the beauty of these formulae seems to me to be comparable with that of the Rogers-Ramanujan identities. So far as I know, nobody else has discovered any formulae which approach them even remotely . . . .*

But Ramanujan went beyond these formulae! To see how, let us first state another identity among the set of forty:

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\varphi(q)}{(q^2;q^2)_\infty} = (-q;q^2)_\infty^2, \tag{4.2}$$

where $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$ is the Jacobi theta function. Regarding (4.2), Andrews [38, p. xxi] says,

*This sort of identity has always appeared to me to lie totally within the realm of modular functions and to be completely resistant to q-series generalization. One of the greatest shocks I got from the Lost Notebook was the following assertion...*

The assertion referred to in the above quote of Andrews is the special case $b = 1$ and $q \to q^4$ of the following exquisitely beautiful identity occurring on page 27 of Ramanujan's Lost Notebook [38], and valid for $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$:

$$\sum_{m=0}^{\infty} \frac{a^{-2m}q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1}q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$

$$= \frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b)\sum_{n=1}^{\infty} a^n q^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \tag{4.3}$$

This multi-parameter identity not only gives (4.2) as a special case when we let $a = b = 1$, replace $q$ by $q^4$, and then use Rogers' identities

$$G(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}, \qquad H(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n},$$

but also an identity connecting $G(q)$ and $H(q)$ with certain fifth order mock theta functions of Ramanujan [13, p. 28].

This is well documented in Andrews [13] where (4.3) was proved by showing that the coefficients of $a^N$, $-\infty < N < \infty$, in the Laurent series expansions of both sides are equal. The present author and Kumar [29] recently generalized (4.3) and gave a compelling evidence as to how Ramanujan may have arrived at this generalized modular relation. In Section 6 of the same paper, they showed that (4.3) and another identity on page 26 of the Lost Notebook, namely [29, Equation (6.1)], together give yet another among the forty identities of Ramanujan.

In [22, p. 73], Birch says,

*'... They support the view that Ramanujan's insight into the arithmetic of modular forms was even greater than has been realized.'*

The word *'They'* refers to the various manuscripts of Ramanujan transcribed by Watson which contain the one having the forty identities for $G(q)$ and $H(q)$. Also, after finding two algebraic relations between these two functions in [36], Ramanujan said, '*Each of these formulae is the simplest of a large class*'. It is believed that the *'large class'* referred to by Ramanujan is the set of forty identities. However, in light of the fact that (4.3) and the other multi-parameter identity on page 26 of the Lost Notebook lead to certain identities among the forty as corollaries, it was speculated in [29] whether Ramanujan intended the term *'large class'* to mean identities of the form in (4.3). This definitely calls for serious research in this direction.

In any case, the identities such as (4.3) show Ramanujan's leaps of imagination transcended the horizon of the modular landscape.

## 5. Concluding remarks

The aim of this article was to introduce an uninitiated reader to the magnificent world of partitions through the lens of Rogers-Ramanujan identities and to delineate two of the multitudinous aspects of the Rogers-Ramanujan functions. Being such a fundamental construct, partitions penetrate almost every branch of mathematics and mathematical sciences. Likewise, the Rogers-Ramanujan identities seem to be universal with more and more seemingly distant areas of Mathematics discovering their presence and importance.

The beautiful and comprehensive text on partition theory is Andrews' *Theory of Partitions* [12]. For an elementary introduction to partitions, the reader is encouraged to read the book by Andrews and Eriksson [17] which contains a nice exposition of several interesting topics in the theory, for example, *'Discovering the first Rogers-Ramanujan identity'*. Another book on the subject which may act as a supplement to [12] is the one by Agarwal, Padmavathamma and Subbarao [2]. The only text written on the Rogers-Ramanujan identities, and a must-read, is the book by Sills [44].

On the other hand, to see the proofs of the forty identities for $G(q)$ and $H(q)$ together in one place, and that too, in the spirit of Ramanujan, the reader is referred to the monograph [21]. An excellent book to know Ramanujan's contributions to the Rogers-Ramanujan as well as other continued fractions along with their proofs is Part I of the series of books on Ramanujan's Lost Notebook by Andrews and Berndt [15].

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