RESTRICTED OVERPARTITIONS AND CONCAVE COMPOSITIONS: THEIR MODULARITY AND ASYMPTOTICS

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ABSTRACT. In this paper we study restricted overpartitions and concave compositions. Using q-series transformations, we show that their generating functions are related to modular forms, mock theta functions, false theta functions, and mock Maass theta functions. Moreover, we obtain their asymptotic main terms. We also study related rank statistics.

1. INTRODUCTION AND STATEMENTS OF RESULTS

A *partition* of $n \in \mathbb{N}_0$ is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{j=1}^{r} \lambda_j = n$. The partition function $p(n)$ counts the number of partitions of n. Its generating function is given by

$$P(q) := \sum_{n \geq 0} p(n) q^n = \frac{1}{(q)_\infty}.$$

Here, and throughout, $|q| < 1$, and for $a \in \mathbb{C}$ and $n \in \mathbb{N}_0 \cup \{\infty\}$, we set $(a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1 - aq^j)$. Note that $P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}$, where $\eta(\tau)$ is the *Dedekind η-function* defined by

$$\eta(\tau) := q^{\frac{1}{24}} (q)_\infty \ \left(q := e^{2\pi i \tau}, \tau \in \mathbb{H}\right).$$

As η is a modular form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ (with multiplier), P is a weakly holomorphic modular form. Hardy and Ramanujan [29] developed the Circle Method, to obtain an asymptotic expansion of $p(n)$ and in the simplest form, its main term equals

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n} \quad \text{as} \quad n \to \infty. \tag{1.1}$$

Dyson [26] introduced the so-called rank of a partition to give a combinatorial explanation of certain congruences of Ramanujan (see [34]) for $p(n)$. The *rank* of a partition is defined as the largest part minus the number of its parts. For $r \in \mathbb{N}$ and $a_1, \ldots, a_r \in \mathbb{C}$, we let $(a_1, \ldots, a_r)_n = (a_1, \ldots, a_r; q)_n = (a_1)_n \ldots (a_r)_n$. The generating function of $N(m,n)$, the number of partitions of n with rank m, is given by (see [11])

$$R(\zeta; q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} N(m,n) \zeta^m q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(\zeta q, \zeta^{-1} q)_n}. \tag{1.2}$$

Note that $f(q) = R(-1;q)$ is one of the mock theta functions introduced by Ramanujan in his last letter to Hardy. For Ramanujan [14], a *mock theta function* is a function F that has asymptotic expansions at every roots of unity[1] of the same type as those of weakly holomorphic modular forms, however there is not a single weakly holomorphic modular form whose asymptotic expansions agree

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[1]Equivalently, considering F as a function of τ, we can also consider asymptotic expansions at rationals.

at all roots of unity with that of F. In his thesis [38], Zwegers "completed" Ramanujan's mock theta functions to harmonic Maass forms[2] by adding additional non-holomorphic terms. Ono and the second author [17] showed that, for roots of unity $\zeta \neq 1$, $R(\zeta; q)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form. Later Berkovich and Garvan [12, Section 5] defined the M_2-rank for partitions without repeated odd parts as the smallest integer at least $\frac{l(\lambda)}{2}$ minus the number of parts of λ, where $l(\lambda)$ is the largest part of λ. The M_2-rank generating function for partitions without repeated odd parts is given by

$$\mathrm{R2}(\zeta; q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} N_2(m,n)\zeta^m q^n = \sum_{n\geq 0} \frac{\left(-q;q^2\right)_n q^{n^2}}{\left(\zeta q^2, \zeta^{-1}q^2; q^2\right)_n}, \tag{1.3}$$

where $N_2(m,n)$ counts the number of partitions of n without repeated odd parts with M_2-rank m (see [32, equation (1.1)]). Similar as for $R(\zeta; q)$, Ono, Rhoades, and the second author [18] showed that if ζ is a root of unity, then $\mathrm{R2}(\zeta; q)$ is a mock modular form. A *mixed mock modular form* is a linear combination of product of modular forms and mock theta functions. An important example comes from concave compositions. A *concave composition* of n is a sum of integers of the form $\sum_{j=1}^{r} a_j + \underline{c} + \sum_{j=1}^{s} b_j = n \;\; \underline{c}, r, s \in \mathbb{N}_0$, where $a_1 \geq a_2 \geq \cdots \geq a_r > \underline{c} < b_s \leq \cdots \leq b_2 \leq b_1$. The distinguished point $\underline{c}$ is the *central part* of the sequence. Let $v(n)$ count the number of concave compositions of n. Its generating function is given by (see [10])

$$V(q) := \sum_{n\geq 0} v(n)q^n = \sum_{n\geq 0} \frac{q^n}{(q^{n+1})_\infty^2}.$$

By [10], V is a mixed mock modular form. We also require the theta function

$$\Theta(-q) := \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \frac{(q)_\infty^2}{(q^2;q^2)_\infty} \tag{1.4}$$

which is a modular form of weight $\frac{1}{2}$. The Fourier coefficients of its reciprocal count overpartitions [23]. An *overpartition* of n is a partition of n in which the first occurrence of a part may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. We denote the generating function of $\overline{p}(n)$ by $\overline{P}$.

False theta functions are theta functions but twisted with sign-factors. Due to this "twist", they are not modular. Now, we give an example of a sequence whose generating function involves a false theta function. A *unimodal sequence* of n is a finite sequence of positive integers a_j $(1 \leq j \leq r)$, b_ℓ $(1 \leq \ell \leq s)$, $r, s \in \mathbb{N}_0$, and $\underline{c} \in \mathbb{N}$ such that $a_r \leq \ldots \leq a_2 \leq a_1 \leq \underline{c} \geq b_1 \geq b_2 \geq \ldots \geq b_s$ for some $k \in \mathbb{N}$ and $\sum_{j=1}^{r} a_j + \underline{c} + \sum_{j=1}^{s} b_j = n$. The generating function of[3] $u(n)$, the number of unimodal sequences of size n, (see [3]) is given by

$$U(q) := \sum_{n\geq 0} u(n)q^n = \sum_{n\geq 0} \frac{q^n}{(q)_n^2} = P^2(q) \sum_{n\geq 0} (-1)^n q^{\frac{n(n+1)}{2}}.$$

Thus U is a *mixed false theta function*, which a linear combinations of false theta functions multiplied by modular forms. Like Jacobi theta functions fit into the theory of Jacobi form, by introducing an extra Jacobi variable, Nazaroglu and the second author [20, Theorem 1.1] showed that the "modular completion" of a two-variable false theta functions transforms like a Jacobi form.

[2]A *harmonic Maass form* is a function that satisfies modularity, is annihilated by the weight k hyperbolic Laplacian, and satisfies a growth condition at the cusps. Its holomorphic part is called a *mock modular form*.

[3]In [3, Section 3], $u(n)$ was denoted by $\sigma\sigma(n)$ and was called stacks with summits of size n.

Next we turn to rank statistics of restricted partitions. Andrews [4] considered partitions into distinct parts with even rank minus the number of those with odd rank. He showed that the corresponding generating function is Ramanujan's σ-function

$$\sigma(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q)_n}. \tag{1.5}$$

Andrews, Dyson, and Hickerson [8] proved that the Fourier coefficients of σ assume all integers infinitely often by relating them to the arithmetic of $\mathbb{Q}(\sqrt{6})$. Cohen [22] then constructed a Maass form from the Fourier coefficients of a linear combination of $\sigma(q)$ and a "companion". These two functions are related to quantum modular forms [37]. A function $f : \mathbb{Q} \setminus S \to \mathbb{R}$ (for some discrete subset S of $\mathbb{Q}$) is a *quantum modular form* if, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the *obstruction to modularity* $f_\gamma(x) := f(x) - (cx+d)^{-k} f(\frac{ax+b}{cx+d})$ is "nice"[4]. Zwegers [39] developed a broader framework by constructing *mock Maass theta functions* which are certain theta functions that are non-modular eigenfunctions of the hyperbolic Laplacian which can be completed to non-holomorphic modular functions.

In this paper, we consider overpartitions of n with the following restrictions:

(1) The smallest part and the even parts are non-overlined,
(2) If the smallest part is odd, then it appears exactly once.

Let $\overline{p}_{\mathrm{od}}(n)$ denote the number of such overpartitions of n. For example, $\overline{p}_{\mathrm{od}}(6) = 7$, since the admissible overpartitions are $6,\ 5+1,\ \overline{5}+1,\ 4+2,\ 3+2+1,\ \overline{3}+2+1,\ 2+2+2$. We have

$$\overline{P}_{\mathrm{od}}(q) := \sum_{n\geq 0} \overline{p}_{\mathrm{od}}(n) q^n = \sum_{n\geq 0} \frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+1}}{\left(q^{2n+2},q^{2n+3};q^2\right)_\infty} + \sum_{n\geq 1} \frac{\left(-q^{2n+1};q^2\right)_\infty q^{2n}}{\left(q^{2n},q^{2n+1};q^2\right)_\infty}. \tag{1.6}$$

Here we show that $\overline{P}_{\mathrm{od}}$ can be written in terms of a false theta function and a mixed mock Maass theta function[5]. For this define[6], following Corson, Favero, Leisinger, and Zubairy [24],

$$W_1(q) := \sum_{n\geq 0} \frac{(-1)^n (q)_n\, q^{\frac{n(n+1)}{2}}}{(-q)_n}. \tag{1.7}$$

Theorem 1.1. *We have*

$$\overline{P}_{\mathrm{od}}(q) = -\frac{(1+q)\left(-q;q^2\right)_\infty}{q\,(q)_\infty} W_1(q) + \frac{(1+q)\left(-q;q^2\right)_\infty}{q\,(q)_\infty} - \frac{1+q}{q} \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{\frac{n(3n-1)}{2}}.$$

Using Theorem 1.1 we obtain the following asymptotic main term for $\overline{p}_{\mathrm{od}}(n)$.

Corollary 1.2. *We have, as $n \to \infty$,*

$$\overline{p}_{\mathrm{od}}(n) \sim \frac{5\pi}{48\sqrt{2}n^{\frac{3}{2}}} e^{\pi\sqrt{\frac{5n}{6}}}.$$

We next consider overpartitions without repeated odd parts with the additional restrictions:

(1) The odd parts are non-overlined.
(2) If the smallest part is even and occurs only once, then it is also non-overlined.

[4]In typical examples, f_γ extends to a real-analytic function on $\mathbb{R}$ except for a finite set of points.

[5]*Mixed mock Maass theta functions* are linear combinations of products of modular forms and mock Maass theta functions.

[6]This is a mock Maass theta function by work of Zwegers (see the discussion at the end of Section 6 of [39]). Similar to the case of σ, the Fourier coefficients of W_1 are related to the arithmetic of $\mathbb{Q}(\sqrt{2})$.

Let $\overline{p}_{\mathrm{ev}}(n)$ denote the number of such weighted overpartitions of n. We have $\overline{p}_{\mathrm{ev}}(6) = 8$, since the admissible overpartitions are $6,\ 5+1,\ 4+2,\ \overline{4}+2,\ 3+2+1,\ 3+\overline{2}+1,\ 2+2+2,\ \overline{2}+2+2$. It is not difficult to see that the generating function of $\overline{p}_{\mathrm{ev}}(n)$ is given by

$$\overline{P}_{\mathrm{ev}}(q) := \sum_{n\geq 1} \overline{p}_{\mathrm{ev}}(n) q^n = \sum_{n\geq 0} \frac{\left(-q^{2n+2}, -q^{2n+3}; q^2\right)_\infty q^{2n+1}}{(q^{2n+2}; q^2)_\infty} + \sum_{n\geq 1} \frac{\left(-q^{2n}, -q^{2n+1}; q^2\right)_\infty q^{2n}}{(q^{2n}; q^2)_\infty}. \tag{1.8}$$

The following theorem shows that $\overline{P}_{\mathrm{ev}}$ is a mixed mock modular form. To be more precise, consider Ramanujan's third order mock theta function

$$\phi(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2; q^2)_n}. \tag{1.9}$$

Theorem 1.3. *We have*

$$\overline{P}_{\mathrm{ev}}(q) = \frac{1+q}{2(q)_\infty}\left(1 - \frac{(q)_\infty^2}{(-q)_\infty^2}\right) + (1+q)\left(\phi(-q) - 1\right).$$

As an application of Theorem 1.3, we determine the asymptotic main term of $\overline{p}_{\mathrm{ev}}(n)$.

Corollary 1.4. *We have, as $n \to \infty$,*

$$\overline{p}_{\mathrm{ev}}(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}.$$

As an application, we obtain asymptotics of $g(n)$, where $g(n)$ counts certain restricted colored partitions studied by Andrews and Kumar [9] whose generating function is given by

$$\sum_{n\geq 0} g(n) q^n := \sum_{n\geq 0} \frac{q^{2n+1}}{(q^{2n+1})_\infty (q^{2n+2}; q^2)_n}.$$

Using this, [9, Theorem 6.1], Theorem 1.3, and the facts that $g(0) = 0$ and $g(1) = 1$, we obtain

$$g(n) + \overline{p}_{\mathrm{ev}}(n-1) = 2p(n-1). \tag{1.10}$$

Employing this, Corollary 1.4, and (1.1), we have the following.

Corollary 1.5. *We have, as $n \to \infty$,*

$$g(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n}.$$

We also give a refinement of $\overline{p}_{\mathrm{od}}(n)$. Let $\overline{p}_{\mathrm{od}}(m,n)$ count those overpartitions which are also counted by $\overline{p}_{\mathrm{od}}(n)$, where the total number of even parts is m counted in the following way. If the smallest part of λ is odd, then m counts the number of even parts and if the smallest part of λ is even, then m counts the number of even parts minus one. For example, taking $n = 6$, we have:

λ	6	$5+1$	$\overline{5}+1$	$4+2$	$3+2+1$	$\overline{3}+2+1$	$2+2+2$
m	0	0	0	1	1	1	2

Thus $\overline{p}_{\mathrm{od}}(0,6) = 3$, $\overline{p}_{\mathrm{od}}(1,6) = 3$, and $\overline{p}_{\mathrm{od}}(2,6) = 1$. We have

$$\overline{P}_{\mathrm{od}}(\zeta; q) := \sum_{m,n\geq 0} \overline{p}_{\mathrm{od}}(m,n) \zeta^m q^n = \sum_{n\geq 0} \frac{\left(-q^{2n+3}; q^2\right)_\infty q^{2n+1}}{(\zeta q^{2n+2}, q^{2n+3}; q^2)_\infty} + \sum_{n\geq 1} \frac{\left(-q^{2n+1}; q^2\right)_\infty q^{2n}}{(\zeta q^{2n}, q^{2n+1}; q^2)_\infty}. \tag{1.11}$$

In Theorem 1.6, we give a new representation of $\overline{P}_{\mathrm{od}}(\zeta;q)$ such that for special values of ζ, we see interesting modular properties. To state our result, we define an extension of the third order mock theta function[7] (see [21])

$$f\left(\zeta_1,\zeta_2;q\right):=\sum_{n\geq 0}\frac{\zeta_1^n\zeta_2^{2n}q^{n^2-3n}}{\left(-\zeta_2,-\frac{\zeta_1\zeta_2}{q}\right)_n}. \tag{1.12}$$

From [21], for $k,\ell\in\mathbb{N}$, $f(q^k,q^\ell;q)$ is a mock theta function in the sense of Ramanujan.

Theorem 1.6. *We have*

$$\overline{P}_{\mathrm{od}}(\zeta;q)=\frac{(1+q)\left(-q^3;q^2\right)_\infty}{\zeta q\left(\zeta q^2,q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{\left(-\frac{1}{q};q^2\right)_n\zeta^n q^{n(n+1)}}{\left(-\zeta q,-q^2;q^2\right)_n}-\frac{1+q}{\zeta q}f\left(\zeta q,q^2;q^2\right)+\frac{1+q}{\zeta q}.$$

For the special case $\zeta=q$ we recall Ramanujan's third order mock theta function

$$f(q):=\sum_{n\geq 0}\frac{q^{n^2}}{(-q)_n^2} \tag{1.13}$$

and a second order mock theta function by McIntosh [33]

$$\mu(q):=\sum_{n\geq 0}\frac{(-1)^n\left(q;q^2\right)_n q^{n^2}}{\left(-q^2;q^2\right)_n^2}. \tag{1.14}$$

The following corollary shows that $\overline{P}_{\mathrm{od}}(q;q)$ is a mixed mock modular form.

Corollary 1.7. *We have*

$$\overline{P}_{\mathrm{od}}(q;q)=\frac{(1+q)\left(-q;q^2\right)_\infty}{q^2\left(q;q^2\right)_\infty^2}\left(-1-q+\mu(-q)\right)-\frac{1+q}{q^2}\left(f\left(q^2\right)-1\right).$$

Next, we let $\zeta\to q^{-2}$ in Theorem 1.6. To state the result, we consider overpartitions of n with the following additional restrictions:

(1) The smallest part is even and even parts are non-overlined.

(2) The difference between odd parts (if they appear) and the smallest even part is at least three.

Let $\overline{p}_{\mathrm{od}}^{[1]}(n)$ denote the number of such overpartitions of n. For example, $\overline{p}_{\mathrm{od}}^{[1]}(9)=4$, since the admissible partitions are $7+2,\ \overline{7}+2,\ 5+2+2,\ \overline{5}+2+2$. We define

$$\overline{P}_{\mathrm{od}}^{[1]}(q):=\sum_{n\geq 2}\overline{p}_{\mathrm{od}}^{[1]}(n)q^n=\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n}}{\left(q^{2n},q^{2n+3};q^2\right)_\infty}. \tag{1.15}$$

Then $\overline{P}_{\mathrm{od}}^{[1]}$ is a linear combination of a mixed mock Maass theta function and a false theta function.

Theorem 1.8. *We have*

$$\overline{P}_{\mathrm{od}}^{[1]}(q)=\frac{q\left(-q^3;q^2\right)_\infty}{(q)_\infty}W_1(q)+\frac{\left(1-2q-q^2\right)\left(-q^3;q^2\right)_\infty}{(1+q)(q)_\infty}+\frac{q}{1+q}\sum_{n\in\mathbb{Z}}\mathrm{sgn}(n)q^{\frac{n(3n-1)}{2}}-\frac{1}{1+q}.$$

We also study a two-variable generalization of $\overline{P}_{\mathrm{ev}}(q)$, denoted by $\overline{P}_{\mathrm{ev}}(\zeta;q)$. Let $\overline{p}_{\mathrm{ev}}(m,n)$ count the number of restricted overpartitions of n enumerated by $\overline{p}_{\mathrm{ev}}(n)$ such that m counts the odd parts plus the non-overlined even parts. For example, taking $n=6$, we have

[7] Note that we consider $f(\zeta_1,\zeta_2;q)$ as a meromorphic function which may have poles. These poles turn out to be important in the proof of Corollary 1.11.

λ	6	$5+1$	$4+2$	$\overline{4}+2$	$3+2+1$	$3+\overline{2}+1$	$2+2+2$	$\overline{2}+2+2$
m	1	2	2	1	3	2	3	2

Therefore $\overline{p}_{\mathrm{ev}}(1,6)=2$, $\overline{p}_{\mathrm{ev}}(2,6)=4$, and $\overline{p}_{\mathrm{ev}}(3,6)=2$. We have

$$\begin{aligned}\overline{P}_{\mathrm{ev}}(\zeta;q) &:= \sum_{m,n\geq 0} \overline{p}_{\mathrm{ev}}(m,n)\zeta^m q^n \\ &= \sum_{n\geq 0} \frac{\left(-q^{2n+2},-\zeta q^{2n+3};q^2\right)_\infty \zeta q^{2n+1}}{\left(\zeta q^{2n+2};q^2\right)_\infty} + \sum_{n\geq 1} \frac{\left(-q^{2n},-\zeta q^{2n+1};q^2\right)_\infty \zeta q^{2n}}{\left(\zeta q^{2n};q^2\right)_\infty}. \end{aligned} \tag{1.16}$$

We recall another (see [21, p. 347]) generalized[8] third order mock theta function

$$\nu\left(\zeta_1,\zeta_2;q\right) := \sum_{n\geq 0} \frac{\zeta_2^{2n} q^{n(n-1)}}{\left(-\frac{\zeta_1^2\zeta_2^2}{q^3};q^2\right)_{n+1}}. \tag{1.17}$$

We decompose $\overline{P}_{\mathrm{ev}}(\zeta;q)$ into three pieces: one comes from ν, another is essentially a modular form, and the third one has modular properties for special values of ζ.

Theorem 1.9. *We have*

$$\begin{aligned}\overline{P}_{\mathrm{ev}}(\zeta;q) = \frac{\zeta q\left(q^2,\zeta q^3,\frac{1}{\zeta q};q^2\right)_\infty}{\left(-q^3,-\zeta q^2;q^2\right)_\infty} + \frac{(1+q)\left(-q^2,-\zeta q;q^2\right)_\infty}{\left(\zeta q^2;q^2\right)_\infty} \sum_{n\geq 0} \frac{\left(-\frac{1}{\zeta};q^2\right)_n (-1)^n q^n}{\left(-q^2;q^2\right)_n} \\ + (1+q)\left((1+\zeta)\,\nu\left(iq, i\left(\zeta q\right)^{\frac{1}{2}};q\right)-1\right).\end{aligned}$$

Next, recall Ramanujan's third order mock theta function

$$\nu(q) := \sum_{n\geq 0} \frac{q^{n(n+1)}}{\left(-q;q^2\right)_{n+1}}. \tag{1.18}$$

The following corollary shows that $\overline{P}_{\mathrm{ev}}(-q;q)$ is a mixed mock modular form.

Corollary 1.10. *We have*

$$\overline{P}_{\mathrm{ev}}(-q;q) = -1-q-\frac{2\left(-q^2;q^4\right)_\infty\left(q^8;q^8\right)_\infty}{\left(q^6;q^4\right)_\infty} + \frac{2\left(q^4;q^4\right)_\infty}{\left(-q^3;q^2\right)_\infty} + \left(1-q^2\right)\nu(-q).$$

Moreover, $\overline{P}_{\mathrm{ev}}(-1;q)$ is basically a modular form.

Corollary 1.11. *We have*

$$\overline{P}_{\mathrm{ev}}(-1;q) = (1+q)\left(\left(q;q^2\right)_\infty - 1\right).$$

Letting $\zeta\to q^{-2}$ in Theorem 1.9, we obtain the following identity of Ramanujan [7].

Corollary 1.12. *We have*

$$2\phi(-q)-f(q) = \frac{\Theta^2(-q)}{(q)_\infty}.$$

Remark 1.13. *There are at least three known proofs of Corollary 1.12, namely, Watson's* [35], *Fine's* [27], *and one in* [7]. *Neither of these explicitly appear in Ramanujan's work which suggests that his own method was perhaps different. Our techniques, however, mainly employ the identities of, or well-known to, Ramanujan, as can be seen in our proof of Corollary 1.12.*

[8]From [21, p. 347], we know that for $k,\ell\in\mathbb{N}$, $\nu(q^k,q^\ell;q)$ is a mock theta function in the sense of Ramanujan.

Finally, we consider concave compositions with the following restrictions:

(1) The central part is even and all other parts are odd.
(2) Odd parts on the left side of the central part may be overlined whereas odd parts to the right of the central part are non-overlined.

Let $\overline{v}_{\mathrm{od}}(n)$ denote the number of such compositions of n. For example, $\overline{v}_{\mathrm{od}}(2)=6$, enumerated by $1+1+\underline{0}$, $\overline{1}+1+\underline{0}$, $1+\underline{0}+1$, $\overline{1}+\underline{0}+1$, $\underline{0}+1+1$, $\underline{2}$. The generating function of $\overline{v}_{\mathrm{od}}(n)$ is given by

$$\overline{V}_{\mathrm{od}}(q) := \sum_{n\geq 0} \overline{v}_{\mathrm{od}}(n)q^n = \sum_{n\geq 0} \frac{\left(-q^{2n+1};q^2\right)_\infty q^{2n}}{\left(q^{2n+1};q^2\right)_\infty^2}.$$

For such a concave composition λ, the *rank relative to the non-overlined odd parts* is defined as $\mathcal{R}(\lambda):=s-r$, where s (resp. r) is the number of non-overlined odd parts to the right (resp. to the left) of the central part of the composition. Let $\overline{v}_{\mathrm{od}}(m,n)$ denote the number of such concave compositions of n with rank with respect to non-overlined odd parts is m. For $n=2$, we have:

λ	$1+1+\underline{0}$	$\overline{1}+1+\underline{0}$	$1+\underline{0}+1$	$\overline{1}+\underline{0}+1$	$\underline{0}+1+1$	$\underline{2}$
$\mathcal{R}(\lambda)=m$	-2	-1	0	1	2	0

Therefore, $\overline{v}_{\mathrm{od}}(-2,2)=\overline{v}_{\mathrm{od}}(2,2)=1$, $\overline{v}_{\mathrm{od}}(-1,2)=\overline{v}_{\mathrm{od}}(1,2)=1$, and $\overline{v}_{\mathrm{od}}(0,2)=2$. The generating function of $\overline{v}_{\mathrm{od}}(m,n)$ is given by

$$\overline{V}_{\mathrm{od}}(\zeta;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \overline{v}_{\mathrm{od}}(m,n)\zeta^m q^n = \sum_{n\geq 0} \frac{\left(-q^{2n+1};q^2\right)_\infty q^{2n}}{\left(\zeta q^{2n+1},\zeta^{-1}q^{2n+1};q^2\right)_\infty}. \tag{1.19}$$

From the following theorem, we see that $\overline{V}_{\mathrm{od}}(\zeta;q)$ is related to known rank generating functions.

Theorem 1.14. *We have*

$$q\overline{V}_{\mathrm{od}}(\zeta;q) = \frac{\left(-q;q^2\right)_\infty \left(\mathrm{R2}\left(-\zeta;q\right)-1\right)}{\left(\zeta q,\zeta^{-1}q;q^2\right)_\infty} - R\left(-\zeta;q^2\right)+1.$$

The paper is organized as follows. In Section 2, first we recall q-series transformations required for this paper and state Ingham's Tauberian theorem and the Euler–Maclaurin summation formula. We prove Theorem 1.1 and Corollary 1.2 in Section 3. Section 4 presents proofs of Theorem 1.3 and Corollary 1.4. In Section 5, we show Theorem 1.6, Corollary 1.7, Theorems 1.8 and 1.9, Corollaries 1.10, 1.11, and 1.12, and Theorem 1.14. Finally in Section 6, we state problems for future research.

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2. Preliminaries

2.1. **q-series identities.** Here we list q-series identities which are used in the proofs of our theorems. We begin with two identities of Ramanujan. The first was stated in [2, equation $(3.17)_{\mathrm{R}}$].

Lemma 2.1. *We have*

$$\sum_{n\geq 0}\frac{(-aq,-bq)_n\, q^{n+1}}{(-cq)_n}=\frac{c}{ab}\sum_{n\geq 1}\frac{\left(-c^{-1}\right)_n\left(\frac{ab}{c}\right)^n q^{\frac{n(n+1)}{2}}}{\left(\frac{aq}{c},\frac{bq}{c}\right)_n}-\frac{c\,(-aq,-bq)_\infty}{ab\,(-cq)_\infty}\sum_{n\geq 1}\frac{\left(\frac{ab}{c^2}\right)^n q^{n^2}}{\left(\frac{aq}{c},\frac{bq}{c}\right)_n}.$$

The second identity of Ramanujan was given in [2, equation $(3.18)_{\mathrm{R}}$].

Lemma 2.2. *We have*

$$\sum_{n\geq 0}\frac{(cq)_n q^n}{(-aq,-bq)_n}=\left(1+a^{-1}\right)\sum_{n\geq 0}\frac{(cq)_n\left(-\frac{b}{a}\right)^n q^{\frac{n(n+1)}{2}}}{\left(-\frac{cq}{a}\right)_{n+1}(-bq)_n}-\frac{a^{-1}(cq)_\infty}{(-aq,-bq)_\infty}\sum_{n\geq 0}\frac{\left(-\frac{b}{a}\right)^n q^{\frac{n(n+1)}{2}}}{\left(-\frac{cq}{a}\right)_{n+1}}.$$

Recall the following two identities from [5, equation (2.2.5) and equation (2.2.6)].

Lemma 2.3. *For $|t|,|q|<1$, we have*

$$\sum_{n\geq 0}\frac{t^n}{(q)_n}=\frac{1}{(t)_\infty}\quad \text{and}\quad \sum_{n\geq 0}\frac{t^n q^{\frac{n(n-1)}{2}}}{(q)_n}=(-t)_\infty\,.$$

Next, we require an identity due to Garvan [28, Theorem 1.2].

Lemma 2.4. *We have*

$$\sum_{n\geq 1}\frac{(-1)^{n+1}\zeta^n q^{n^2}}{\left(1-\zeta q^{2n}\right)\left(\zeta q;q^2\right)_n}=\sum_{n\geq 1}\frac{(q)_{n-1}\,\zeta^n q^{\frac{n(n+1)}{2}}}{(\zeta q)_n}.$$

Due to Kang [31, equation (7.5)], we have the following:

Lemma 2.5. *We have*

$$\sum_{n\geq 0}\frac{q^{\frac{n(n+1)}{2}}}{(-q)_{n+1}}=1.$$

In [25, Theorem 1.1], we find the following identity.

Lemma 2.6. *We have*

$$\sum_{n\geq 0}\frac{nq^{\frac{n(n-1)}{2}}}{(-q)_n}=\sigma(q).$$

From [6, equation (2.3.1)], we have the following lemma.

Lemma 2.7. *We have, for $a\in\mathbb{C}$ and $|bq|<1$,*

$$\sum_{n\geq 0}\frac{(ab)^n q^{n^2}}{(aq,bq)_n}=1+a\sum_{n\geq 1}\frac{(bq)^n}{(aq)_n}.$$

From [5, Corollary 2.4], we record the following identity.

Lemma 2.8. *For $|c|<|ab|,|q|<1$, we have*

$$\sum_{n\geq 0}\frac{(a,b)_n\left(\frac{c}{ab}\right)^n}{(q,c)_n}=\frac{\left(\frac{c}{a},\frac{c}{b}\right)_\infty}{\left(c,\frac{c}{ab}\right)_\infty}.$$

From [5, Example 10, p. 29], we record the following identity.

Lemma 2.9. *We have*

$$1+\sum_{n\geq 1}(-1)^n\left(\zeta^{3n-1}q^{\frac{n(3n-1)}{2}}+\zeta^{3n}q^{\frac{n(3n+1)}{2}}\right)=\sum_{n\geq 0}\frac{(-1)^n\zeta^{2n}q^{\frac{n(n+1)}{2}}}{(\zeta q)_n}.$$

We also require identities by Bhoria, Eyyunni, and Maji. The first is (see [15, Theorem 2.1]).

Lemma 2.10. *We have, for* $|ad|$, $|cq|<1$,

$$\sum_{n\geq 1}\frac{\left(\frac{b}{a},\frac{c}{d}\right)_n(ad)^n}{(b,cq)_n}=\frac{(a-b)(d-c)}{ad-b}\sum_{n\geq 0}\frac{\left(a,\frac{bd}{c}\right)_n c^n}{(b,ad)_n}\left(\frac{adq^n}{1-adq^n}-\frac{bq^n}{1-bq^n}\right).$$

The second identity (see [15, Theorem 2.2]) is the following:

Lemma 2.11. *For* $|cq|<1$,

$$\sum_{n\geq 1}\frac{\left(\frac{c}{d}\right)_n(-\zeta d)^n q^{\frac{n(n+1)}{2}}}{(\zeta q,cq)_n}=\frac{\zeta(c-d)}{c}\sum_{n\geq 1}\frac{\left(\frac{\zeta dq}{c}\right)_{n-1}(cq)^n}{(\zeta q)_n}.$$

Next, we list the following identity from [13, equation (3.2.6)].

Lemma 2.12. *We have*

$$1+4\sum_{n\geq 0}\frac{(-1)^{n+1}q^{2n+1}}{1+q^{2n+1}}=\frac{(q)_\infty^2}{(-q)_\infty^2}.$$

Using [1, equation (4.5)] with $q\mapsto q^2, \alpha=-q, \beta=-\zeta q^3$, and $t=-\zeta q^2$, we get the following.

Lemma 2.13. *We have*

$$\sum_{n\geq 0}\frac{(-q;q^2)_n(-1)^n\zeta^n q^{2n}}{(-\zeta q^3;q^2)_n}=\frac{(q,\zeta q^2)_\infty\left(\frac{1}{\zeta q};q^2\right)_\infty}{2(-\zeta q^2)_\infty}+\frac{1}{2}\left(1+\frac{1}{\zeta q}\right)\sum_{n\geq 0}\frac{\left(-\frac{1}{\zeta};q^2\right)_n(-1)^n q^n}{(-q^2;q^2)_n}.$$

Finally, we require the Rogers–Fine identity [27, equation (14.1)].

Lemma 2.14. *We have*

$$\sum_{n\geq 0}\frac{(\alpha)_n t^n}{(\beta)_n}=\sum_{n\geq 0}\frac{\left(\alpha,\frac{\alpha tq}{\beta}\right)_n\left(1-\alpha tq^{2n}\right)\left(\frac{\beta t}{q}\right)^n q^{n^2}}{(\beta)_n(t)_{n+1}}.$$

2.2. **A Tauberian theorem.** We also require a modified version of Ingham's Tauberian Theorem [30], stated in [19, Theorem 1.1]. For $\Delta\geq 0$ define

$$R_\Delta:=\{x+iy:\, x,y\in\mathbb{R},\, x>0,\, |y|\leq \Delta x\}.$$

Proposition 2.15. *Let* $B(q)=\sum_{n\geq 0}b(n)q^n$ *be a power series whose radius of convergence is at least one and assume that the* $b(n)$ *are non-negative and weakly increasing. Also suppose that* λ, β, $\gamma\in\mathbb{R}$ *with* $\gamma>0$ *exist such that*

$$B\left(e^{-t}\right)\sim\lambda t^\beta e^{\frac{\gamma}{t}}\quad \text{as } t\to 0^+,\qquad B\left(e^{-z}\right)\ll |z|^\beta e^{\frac{\gamma}{|z|}}\quad \text{as } z\to 0, \tag{2.1}$$

with the latter condition holding in each region R_Δ *for* $\Delta\geq 0$. *Then we have*

$$b(n)\sim\frac{\lambda\gamma^{\frac{\beta}{2}+\frac{1}{4}}}{2\sqrt{\pi}n^{\frac{\beta}{2}+\frac{3}{4}}}e^{2\sqrt{\gamma n}}\qquad \text{as } n\to\infty.$$

Moreover, we require the asymptotic behavior of $(q)_\infty$ as $q\to 1$.

Lemma 2.16. *Let $q = e^{-z}$ and $\Delta \geq 0$. Then, as $z \to 0$ in R_Δ, we have*

$$(q)_\infty \sim \sqrt{\frac{2\pi}{z}} e^{-\frac{\pi^2}{6z}}.$$

2.3. **The Euler–Maclaurin summation formula.** We use a variation of the Euler–Maclaurin summation formula, an important tool for computing asymptotic expansions of infinite sums, studied by Zagier [36, Proposition 3] for one-dimensional smooth functions and extended by Jennings-Shaffer, Mahlburg, and the second author [19, Theorem 1.2] for multi-dimensional complex-valued functions. A function g, defined over an unbounded domain $D \subset \mathbb{C}$, is of *sufficient decay* if there exists $\varepsilon > 0$ such that $g(w) \ll |w|^{-1-\varepsilon}$ uniformly as $|w| \to \infty$ in D. For $0 \leq \theta < \frac{\pi}{2}$, we define $D_\theta := \{re^{i\alpha} : r \geq 0 \text{ and } |\alpha| \leq \theta\}$.

Proposition 2.17. *Let $0 \leq \theta < \frac{\pi}{2}$ and g be a holomorphic function in a domain containing D_θ. Also assume that g and all of its derivatives are of sufficient decay. Then, for $a \in \mathbb{R}_{\geq 0}$, $N \in \mathbb{N}_0$, and as $z \to 0$ uniformly in D_θ, we have*

$$\sum_{m\geq 0} g\left((m+a)z\right) = \frac{1}{z}\int_0^\infty g(w)dw - \sum_{n=0}^{N-1} \frac{B_{n+1}(a)g^{(n)}(0)}{(n+1)!} z^n + O\left(z^N\right),$$

where $B_n(x)$ denotes the n-th Bernoulli polynomial.

Next, we consider two-dimensional sums. Throughout we write $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. A function f defined on an unbounded domain $D \subset \mathbb{C}^2$ is of *sufficient decay* if there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $f(\boldsymbol{x}) \ll |w_1|^{-1-\varepsilon_1}|w_2|^{-1-\varepsilon_2}$ uniformly as $|w_1| + |w_2| \to \infty$ in D. From [19, Proposition 5.2] (with $r = 2$), we have the following.

Proposition 2.18. *Let $0 \leq \theta < \frac{\pi}{2}$ and f be a holomorphic function in a domain containing $D_\theta \times D_\theta$. Also assume that f and all of its derivatives are of sufficient decay. Then for $\boldsymbol{a} \in \mathbb{R}^2_{\geq 0}$, $N \in \mathbb{N}_0$, and as $z \to 0$ uniformly in D_θ, we have*

$$\begin{aligned}\sum_{m\geq 0} f\left((\boldsymbol{m}+\boldsymbol{a})z\right) = {}& \frac{1}{z^2}\int_0^\infty \int_0^\infty f(\boldsymbol{x})dx_1dx_2 - \frac{1}{z}\sum_{n_1=0}^{N} \frac{B_{n_1+1}\left(a_1\right)z^{n_1}}{(n_1+1)!}\int_0^\infty f^{(n_1,0)}\left(0,x_2\right)dx_2 \\ & - \frac{1}{z}\sum_{n_2=0}^{N} \frac{B_{n_2+1}\left(a_2\right)z^{n_2}}{(n_2+1)!}\int_0^\infty f^{(0,n_2)}\left(x_1,0\right)dx_1 \\ & + \sum_{n_1+n_2<N} \frac{B_{n_1+1}\left(a_1\right)B_{n_2+1}\left(a_2\right)f^{(n_1,n_2)}\left(\boldsymbol{0}\right)}{(n_1+1)!\,(n_2+1)!} + O\left(z^N\right).\end{aligned}$$

3. Proof of Theorem 1.1 and Corollary 1.2

In this section, we prove Theorem 1.1 and Corollary 1.2. We start with Theorem 1.1.

Proof of Theorem 1.1. By (1.6), we have

$$\overline{P}_{\text{od}}(q) = \frac{q\left(-q;q^2\right)_\infty}{\left(q^2,q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(q^2,q^3;q^2\right)_n q^{2n}}{\left(-q^3;q^2\right)_n}. \tag{3.1}$$

Changing $q \mapsto q^2$ in Lemma 2.1 and then dividing both sides by q^2, we get

$$\sum_{n\geq 0}\frac{\left(-aq^2,-bq^2;q^2\right)_n q^{2n}}{\left(-cq^2;q^2\right)_n}$$

$$= \frac{c}{abq^2}\sum_{n\geq 1}\frac{\left(-c^{-1};q^2\right)_n\left(\frac{ab}{c}\right)^n q^{n(n+1)}}{\left(\frac{aq^2}{c},\frac{bq^2}{c};q^2\right)_n} - \frac{c\left(-aq^2,-bq^2;q^2\right)_\infty}{abq^2\left(-cq^2;q^2\right)_\infty}\sum_{n\geq 1}\frac{\left(\frac{ab}{c^2}\right)^n q^{2n^2}}{\left(\frac{aq^2}{c},\frac{bq^2}{c};q^2\right)_n}.$$

Setting $a=-1,\ b=-q$, and $c=q$, yields

$$\sum_{n\geq 0}\frac{\left(q^2,q^3;q^2\right)_n q^{2n}}{\left(-q^3;q^2\right)_n} = \frac{1+q}{q^3}\sum_{n\geq 1}\frac{q^{n(n+1)}}{\left(1+q^{2n-1}\right)\left(-q^2;q^2\right)_n} - \frac{\left(q^2,q^3;q^2\right)_\infty}{q^2\left(-q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}}.$$

Plugging this into (3.1), we obtain

$$\overline{P}_{\mathrm{od}}(q) = \frac{(1+q)\left(-q;q^2\right)_\infty}{q^2\left(q^2,q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{q^{n(n+1)}}{\left(1+q^{2n-1}\right)\left(-q^2;q^2\right)_n} - \frac{1+q}{q}\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}}. \tag{3.2}$$

Letting $\zeta=-q^{-1}$ in Lemma 2.4, we get

$$\sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(1+q^{2n-1}\right)\left(-q^2;q^2\right)_{n-1}} = 1+\sum_{n\geq 1}\frac{(-1)^n (q)_n q^{\frac{n(n+1)}{2}}}{(-q)_n}. \tag{3.3}$$

Continuing with the left-hand side of (3.3) and writing $1+q^{2n}=1+q^{2n-1}-(1-q)q^{2n-1}$, we have

$$\sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(1+q^{2n-1}\right)\left(-q^2;q^2\right)_{n-1}} = \sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(-q^2;q^2\right)_n} - \frac{1-q}{q}\sum_{n\geq 1}\frac{q^{n(n+1)}}{\left(1+q^{2n-1}\right)\left(-q^2;q^2\right)_n}. \tag{3.4}$$

Plugging (3.3) into (3.4) and then applying Lemma 2.5 with $q\mapsto q^2$, we obtain

$$\sum_{n\geq 1}\frac{q^{n(n+1)}}{\left(1+q^{2n-1}\right)\left(-q^2;q^2\right)_n} = -\frac{q}{1-q}\sum_{n\geq 1}\frac{(-1)^n (q)_n q^{\frac{n(n+1)}{2}}}{(-q)_n}. \tag{3.5}$$

Combining this with (3.2) and (1.7), we have

$$\overline{P}_{\mathrm{od}}(q) = -\frac{(1+q)\left(-q;q^2\right)_\infty}{q\,(q)_\infty}W_1(q) + \frac{(1+q)\left(-q;q^2\right)_\infty}{q\,(q)_\infty} - \frac{1+q}{q}\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}}.$$

Thus, to conclude the proof, we need to show that

$$\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}} = \sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{\frac{n(3n-1)}{2}}. \tag{3.6}$$

First note that

$$\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{\frac{n(3n-1)}{2}} = \sum_{n\geq 1}\left(1-q^n\right)q^{\frac{n(3n-1)}{2}} = -\sum_{n\geq 1}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(-q)_n},$$

by setting $\zeta=-1$ in Lemma 2.9. To prove (3.6), it thus suffices to show that

$$\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}} = -\sum_{n\geq 1}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(-q)_n}.$$

To show this, we split the left-hand side depending on the parity of n and then use Lemma 2.5. $\square$

We now determine the asymptotics of $\overline{p}_{\mathrm{od}}(n)$ as $n\to\infty$. For this, we prove the monotonicity of $\overline{p}_{\mathrm{od}}(n)$. For $A(q)=\sum_{n\geq 0}a(n)q^n$, by $A(q)\succeq 0$, we mean that $a(n)\geq 0$ for $n\in\mathbb{N}_0$.

Lemma 3.1. *For $n\in\mathbb{N}_0$, we have $\overline{p}_{\mathrm{od}}(n+1)\geq\overline{p}_{\mathrm{od}}(n)$.*

Proof. Proving $\overline{p}_{\mathrm{od}}(n+1) \geq \overline{p}_{\mathrm{od}}(n)$ for $n \in \mathbb{N}_0$ is equivalent to showing that $(1-q)\overline{P}_{\mathrm{od}}(q) \succeq 0$. By (3.1), we have

$$\overline{P}_{\mathrm{od}}(q) = q\,(1+q) \sum_{n \geq 0} \frac{\left(-q^{2n+3}; q^2\right)_\infty q^{2n}}{\left(q^{2n+2}\right)_\infty}.$$

Thus, to prove $(1-q)\overline{P}_{\mathrm{od}}(q) \succeq 0$, it suffices to show that

$$\left(1-q^2\right) \sum_{n \geq 0} \frac{\left(-q^{2n+3}; q^2\right)_\infty q^{2n}}{\left(q^{2n+2}\right)_\infty} = \frac{\left(-q^3; q^2\right)_\infty}{\left(q^3\right)_\infty} + \left(1-q^2\right) \sum_{n \geq 1} \frac{\left(-q^{2n+3}; q^2\right)_\infty q^{2n}}{\left(q^{2n+2}\right)_\infty} \succeq 0. \tag{3.7}$$

Note that $\frac{(-q^3;q^2)_\infty}{(q)_\infty^3} \succeq 0$. Note that (3.7) follows if we show that, for $n \in \mathbb{N}$,

$$\left(1-q^2\right) \frac{\left(-q^{2n+3}; q^2\right)_\infty}{\left(q^{2n+2}\right)_\infty} - 1 + q^2 \succeq 0. \tag{3.8}$$

Indeed using this, we see that

$$\left(1-q^2\right) \sum_{n \geq 1} \frac{\left(-q^{2n+3}; q^2\right)_\infty q^{2n}}{\left(q^{2n+2}\right)_\infty} \succeq \left(1-q^2\right) \sum_{n \geq 1} q^{2n} = q^2 \succeq 0.$$

To prove (3.8), we use Lemma 2.3, to write

$$\begin{aligned}\left(1-q^2\right) \frac{\left(-q^{2n+3}; q^2\right)_\infty}{\left(q^{2n+2}\right)_\infty} - 1 + q^2 &= \left(1-q^2\right) \left(\sum_{r,s \geq 0} \frac{q^{r^2+(2n+2)(r+s)}}{\left(q^2;q^2\right)_r (q)_s} - 1 \right) \\ &= \sum_{r \geq 1} \frac{q^{r^2+(2n+2)r}}{\left(q^4;q^2\right)_{r-1}} + (1+q) \sum_{\substack{r \geq 0 \\ s \geq 1}} \frac{q^{r^2+(2n+2)(r+s)}}{\left(q^2;q^2\right)_r \left(q^2\right)_{s-1}} \succeq 0.\end{aligned}$$

This finishes the proof. $\square$

Proof of Corollary 1.2. We first note that $\overline{p}_{\mathrm{od}}(n)$ is non-negative because it counts restricted overpartitions and by Lemma 3.1, it is weakly increasing. Thus it remains to determine the asymptotics of $\overline{P}_{\mathrm{od}}(q)$. With $q = e^{-z}$ and as $z \to 0$ in R_Δ, we show that

$$\overline{P}_{\mathrm{od}}(q) \sim \frac{z^{\frac{3}{2}} e^{\frac{5\pi^2}{24z}}}{\sqrt{2\pi}}. \tag{3.9}$$

We split $\overline{P}_{\mathrm{od}}(q)$ as in Theorem 1.1. Using Proposition 2.17, the final term can be bounded as

$$-\frac{1+q}{q} \sum_{n \in \mathbb{Z}} \operatorname{sgn}(n) q^{\frac{n(3n-1)}{2}} \ll \frac{1}{\sqrt{z}}.$$

The rest can be written as

$$\frac{(1+q)\left(-q;q^2\right)_\infty}{q(q)_\infty} \left(1 - W_1(q)\right).$$

Using Lemma 2.16, we have, as $z \to 0$ in R_Δ,

$$\frac{(1+q)\left(-q;q^2\right)_\infty}{q(q)_\infty} \sim \sqrt{\frac{2z}{\pi}} e^{\frac{5\pi^2}{24z}}.$$

So to prove (3.9), we need to show that

$$1 - W_1(q) \sim \frac{z}{2}. \tag{3.10}$$

From [24, Theorem 2.3], we have

$$W_1(q)=\sum_{n\geq 0}(-1)^n q^{2n^2+n}\left(1-q^{2n+1}\right)+2\sum_{\substack{n\geq 0\\ 1\leq j\leq n}}(-1)^{n+j}q^{2n^2+n-j^2}\left(1-q^{2n+1}\right). \tag{3.11}$$

The first summand on the right-hand side of (3.11) may be written as

$$\begin{aligned}
&q^{-\frac{1}{8}}\sum_{n\geq 0}\left(q^{2\left(2n+\frac{1}{4}\right)^2}-q^{2\left(2n+\frac{3}{4}\right)^2}-q^{2\left(2n+\frac{5}{4}\right)^2}+q^{2\left(2n+\frac{7}{4}\right)^2}\right)\\
&\qquad=q^{-\frac{1}{8}}\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)\sum_{n\geq 0}q^{8\left(n+\frac{j}{8}\right)^2}=e^{\frac{z}{8}}\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)\sum_{n\geq 0}g\left(\left(n+\frac{j}{8}\right)\sqrt{z}\right),
\end{aligned} \tag{3.12}$$

where $g(x):=e^{-8x^2}$. Applying Proposition 2.17 with $N=3$ and $z\mapsto\sqrt{z}$, we have

$$\begin{aligned}
&\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)\sum_{n\geq 0}g\left(\left(n+\frac{j}{8}\right)\sqrt{z}\right)\\
&\qquad=\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)\left(\frac{1}{\sqrt{z}}\int_0^\infty g(w)dw-\sum_{n=0}^{3}\frac{B_{n+1}\left(\frac{j}{8}\right)g^{(n)}(0)z^{\frac{n}{2}}}{(n+1)!}+O\left(z^2\right)\right)\\
&\qquad=-\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)\sum_{n=0}^{1}\frac{B_{2n+1}\left(\frac{j}{8}\right)g^{(2n)}(0)z^n}{(2n+1)!}+O\left(z^2\right)=O\left(z^2\right).
\end{aligned} \tag{3.13}$$

Here we use that $\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)=0$, note that g is an even function, and compute

$$\sum_{j\in\{1,3,5,7\}}\left(\frac{2}{j}\right)B_{2n+1}\left(\frac{j}{8}\right)=0\quad\text{for } n\in\{0,1\}.$$

Next, shifting $n\mapsto n+j$ and then $j\mapsto j+1$, the second sum in (3.11) becomes

$$\begin{aligned}
&2q^{-\frac{1}{8}}\sum_{n,j\geq 0}(-1)^n\left(q^{2\left(n+\frac{1}{4}\right)^2+4\left(n+\frac{1}{4}\right)(j+1)+(j+1)^2}-q^{2\left(n+\frac{3}{4}\right)^2+4\left(n+\frac{3}{4}\right)(j+1)+(j+1)^2}\right)\\
&\qquad=2q^{-\frac{1}{8}}\sum_{\delta,r\in\{0,1\}}(-1)^{\delta+r}\sum_{n,j\geq 0}q^{2\left(2n+\delta+\frac{1+2r}{4}\right)^2+4\left(2n+\delta+\frac{1+2r}{4}\right)(j+1)+(j+1)^2}\\
&\qquad=2e^{\frac{z}{8}}\sum_{\delta,r\in\{0,1\}}(-1)^{\delta+r}\sum_{n,j\geq 0}f\left(\left(n+\frac{\delta}{2}+\frac{1+2r}{8},j+1\right)\sqrt{z}\right),
\end{aligned}$$

where $f(x,y):=e^{-8x^2-8xy-y^2}$. Applying Proposition 2.18 yields

$$\begin{aligned}
&2e^{\frac{z}{8}}\sum_{\delta,r\in\{0,1\}}(-1)^{\delta+r}\left(\frac{1}{z}\int_0^\infty\int_0^\infty f(x,y)dxdy-\frac{1}{\sqrt{z}}\left(\sum_{n=0}^{N}\frac{B_{n+1}\left(\frac{\delta}{2}+\frac{1+2r}{8}\right)z^{\frac{n}{2}}}{(n+1)!}\int_0^\infty f^{(n,0)}(0,y)dy\right.\right.\\
&\left.+\sum_{m=0}^{N}\frac{B_{m+1}\left(1\right)z^{\frac{m}{2}}}{(m+1)!}\int_0^\infty f^{(0,m)}(x,0)dx\right)+\sum_{n+m<N}\frac{B_{n+1}\left(\frac{\delta}{2}+\frac{1+2r}{8}\right)B_{m+1}(1)}{(n+1)!(m+1)!}f^{(n,m)}(0,0)z^{\frac{n+m}{2}}\\
&\left.\qquad\qquad+O\left(z^{\frac{N}{2}}\right)\right).
\end{aligned}$$

Now those parts cancel that are independent of δ, r because of the $(-1)^{\delta+r}$. Moreover $f^{(n,m)}(0,0) = 0$ unless $n+m$ is even. Thus with $N=4$, the above becomes

$$2e^{\frac{z}{8}} \sum_{\delta, r \in \{0,1\}} (-1)^{\delta+r} \left(-\sum_{n=0}^{4} \frac{B_{n+1}\left(\frac{\delta}{2}+\frac{1+2r}{8}\right) z^{\frac{n-1}{2}}}{(n+1)!} \int_0^\infty f^{(n,0)}(0,y)dy \right.$$
$$\left. + \sum_{\substack{n+m\leq 3 \\ 2|(n+m)}} \frac{B_{n+1}\left(\frac{\delta}{2}+\frac{1+2r}{8}\right) B_{m+1}(1)}{(n+1)!(m+1)!} f^{(n,m)}(0,0) z^{\frac{n+m}{2}} \right) + O\left(z^2\right). \quad (3.14)$$

We compute

$$\sum_{\delta, r \in \{0,1\}} (-1)^{\delta+r} B_{n+1}\left(\frac{\delta}{2}+\frac{1+2r}{8}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{4} & \text{if } n=1, \\ -\frac{11}{128} & \text{if } n=3. \end{cases} \quad (3.15)$$

Note that n odd, $0 \leq n+m \leq 3$, and $2 \,|\, (n+m)$ implies that $n=m=1$. Thus (3.14) becomes

$$e^{\frac{z}{8}} \sum_{\delta, r \in \{0,1\}} (-1)^{\delta+r} \left(-B_2\left(\frac{\delta}{2}+\frac{1+2r}{8}\right) \int_0^\infty f^{(1,0)}(0,y)dy - \frac{B_4\left(\frac{\delta}{2}+\frac{1+2r}{8}\right) z}{12} \int_0^\infty f^{(3,0)}(0,y)dy \right.$$
$$\left. + \frac{B_2\left(\frac{\delta}{2}+\frac{1+2r}{8}\right)}{12} f^{(1,1)}(0,0) z \right) + O\left(z^2\right). \quad (3.16)$$

Computing

$$\int_0^\infty f^{(1,0)}(0,y)dy = -4, \quad \int_0^\infty f^{(3,0)}(0,y)dy = -64, \text{ and } f^{(1,1)}(0,0) = -8,$$

and using (3.15), (3.16) becomes $1 - \frac{z}{2} + O(z^2)$. This implies that

$$2 \sum_{\substack{n \geq 0 \\ 1 \leq j \leq n}} (-1)^{n+j} q^{2n^2+n-j^2} \left(1 - q^{2n+1}\right) = 1 - \frac{z}{2} + O\left(z^2\right).$$

Combining this with (3.13), (3.12), (3.11), and using Proposition 2.15, we obtain the claim. ◻

4. Proof of Theorem 1.3 and Corollary 1.4

First we prove Theorem 1.3. For this, we need the following lemma, which is obtained by using Lemma 2.10 and Lemma 2.12.

Lemma 4.1. *We have*

$$\sum_{n \geq 1} \frac{(-1)^n \left(q^2; q^2\right)_{n-1} q^{n^2}}{(-q)_{2n}} = \frac{1}{4} \left(\frac{(q)_\infty^2}{(-q)_\infty^2} - 1 \right).$$

Proof of Theorem 1.3. Using (1.8), we have

$$\overline{P}_{\mathrm{ev}}(q) = \frac{q}{(q)_\infty} \sum_{n \geq 0} \frac{\left(q^2; q^2\right)_n q^{2n}}{(-q^2, -q^3; q^2)_n}. \quad (4.1)$$

Changing $q \mapsto q^2$ and then setting $a = c = 1$ and $b = q$ in Lemma 2.2, we get

$$\sum_{n \geq 0} \frac{\left(q^2; q^2\right)_n q^{2n}}{(-q^2, -q^3; q^2)_n} = 2 \sum_{n \geq 0} \frac{(-1)^n \left(q^2; q^2\right)_n q^{n^2+2n}}{(-q^2; q^2)_{n+1} (-q^3; q^2)_n} - (1+q)(q)_\infty \sum_{n \geq 0} \frac{(-1)^n q^{n^2+2n}}{(-q^2; q^2)_{n+1}}.$$

Applying this to (4.1), using (1.9) and then invoking Lemma 4.1, we conclude the proof. $\square$

Next, we determine the asymptotics of $\overline{p}_{\mathrm{ev}}(n)$. For this, we prove its monotonicity.

Lemma 4.2. *We have, for $n \in \mathbb{N}_0$, $\overline{p}_{\mathrm{ev}}(n) \leq \overline{p}_{\mathrm{ev}}(n+1)$.*

Proof. By (4.1), we have

$$(1-q)\,\overline{P}_{\mathrm{ev}}(q) = \frac{q}{(q^2)_\infty} + \left(1-q^2\right)\sum_{n\geq 0}\frac{\left(-q^{2n+4}\right)_\infty q^{2n+3}}{\left(q^{2n+4};q^2\right)_\infty}. \tag{4.2}$$

To prove the monotonicity of $\overline{p}_{\mathrm{ev}}(n)$, we need to show that $(1-q)\overline{P}_{\mathrm{ev}}(q) \succeq 0$. Since $\frac{q}{(q^2)_\infty} \succeq 0$, by (4.2), this follows if we show that for $n \in \mathbb{N}_0$,

$$\left(1-q^2\right)\frac{\left(-q^{2n+4}\right)_\infty}{\left(q^{2n+4};q^2\right)_\infty} - 1 + q^2 \succeq 0.$$

To prove this, using Lemma 2.3, we have

$$\begin{aligned}\left(1-q^2\right)\frac{\left(-q^{2n+4}\right)_\infty}{\left(q^{2n+4};q^2\right)_\infty} - 1 + q^2 &= \left(1-q^2\right)\left(\sum_{r,s\geq 0}\frac{q^{\frac{r(r-1)}{2}+(2n+4)(r+s)}}{(q)_r\left(q^2;q^2\right)_s} - 1\right)\\ &= (1+q)\sum_{r\geq 1}\frac{q^{\frac{r(r-1)}{2}+(2n+4)r}}{\left(q^2\right)_{r-1}} + \sum_{\substack{r\geq 0\\ s\geq 1}}\frac{q^{\frac{r(r-1)}{2}+(2n+4)(r+s)}}{(q)_r\left(q^4;q^2\right)_{s-1}} \succeq 0.\end{aligned}$$

This proves the claim. $\square$

Proof of Corollary 1.4. By Lemma 4.2, $\overline{p}_{\mathrm{ev}}(n)$ is non-negative and weakly increasing. Therefore, to apply Proposition 2.15, it remains to verify that $\overline{P}_{\mathrm{ev}}(q)$ satisfies (2.1). We show that

$$\overline{P}_{\mathrm{ev}}(q) \sim \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}} \quad \text{with } q = e^{-z}, \text{ as } z \to 0 \text{ in } R_\Delta. \tag{4.3}$$

Using Lemma 2.16, we have, as $z \to 0$ in R_Δ,

$$\frac{1+q}{2(q)_\infty}\left(1 - \frac{(q)_\infty^2}{(-q)_\infty^2}\right) \sim \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}.$$

Thus, by Theorem 1.3, (4.3) follows if we show that

$$(1+q)\left(\phi(-q)-1\right) \ll e^{\frac{\pi^2}{24z}}.$$

This indeed holds by [16] and Lemma 2.16. Applying Proposition 2.15, we conclude the proof. $\square$

5. Proof of Theorems 1.6, 1.8, 1.9, and 1.14

Proof of Theorem 1.6. Using the generating function (1.11), and then applying Lemma 2.1 with $q \mapsto q^2$, $a = -\zeta$, $b = -q$, and $c = q$, we obtain

$$\begin{aligned}P_{\mathrm{od}}(\zeta;q) &= \frac{(1+q)\left(-q^3;q^2\right)_\infty}{\zeta q\left(\zeta q^2, q^3;q^2\right)_\infty}\\ &\quad\times\left(\sum_{n\geq 1}\frac{\left(-\frac{1}{q};q^2\right)_n \zeta^n q^{n(n+1)}}{\left(-\zeta q, -q^2;q^2\right)_n} - \frac{\left(\zeta q^2, q^3;q^2\right)_\infty}{\left(-q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{\zeta^n q^{2n^2-n}}{\left(-\zeta q,-q^2;q^2\right)_n}\right).\end{aligned} \tag{5.1}$$

Plugging in (1.12) with $q \mapsto q^2, \zeta_1 = \zeta q$, and $\zeta_2 = q^2$ yields the theorem. $\square$

Proof of Corollary 1.7. Substituting $\zeta = q$ into the second identity of Theorem 1.6, using the definition of $f(\zeta_1, \zeta_2; q)$ from (1.12) with $q \mapsto q^2$, $\zeta_1 = \zeta_2 = q^2$, and (1.13) with $q \mapsto q^2$, we get

$$\overline{P}_{\mathrm{od}}(q;q) = \frac{(1+q)\left(-q;q^2\right)_\infty}{q^3\left(q^3;q^2\right)_\infty^2} \sum_{n\geq 1} \frac{\left(-q;q^2\right)_{n-1} q^{n^2+2n}}{\left(-q^2;q^2\right)_n^2} - \frac{1+q}{q^2}\left(f\left(q^2\right) - 1\right).$$

To complete the proof, we need to show that

$$\sum_{n\geq 1} \frac{\left(-q;q^2\right)_{n-1} q^{n^2+2n}}{\left(-q^2;q^2\right)_n^2} = \frac{q}{(1-q)^2}\left(-1-q+\mu(-q)\right). \tag{5.2}$$

Applying Lemma 2.11 with $q \mapsto q^2, \zeta = c = -1$, and $d = q$, we have

$$\sum_{n\geq 1} \frac{\left(-\frac{1}{q};q^2\right)_n q^{n^2+2n}}{\left(-q^2;q^2\right)_n^2} = -\frac{1+q}{1-q}\sum_{n\geq 1}\frac{\left(q;q^2\right)_n(-1)^n q^{2n}}{\left(-q^2;q^2\right)_n}.$$

Therefore, we obtain

$$\sum_{n\geq 1} \frac{\left(-q;q^2\right)_{n-1} q^{n^2+2n}}{\left(-q^2;q^2\right)_n^2} = \frac{q}{(1-q)^2}\sum_{n\geq 1}\frac{\left(\frac{1}{q};q^2\right)_{n+1}(-1)^n q^{2n+1}}{\left(-q^2;q^2\right)_n}.$$

Thus, following (1.14), to prove (5.2), it remains to show that

$$\sum_{n\geq 1}\frac{\left(\frac{1}{q};q^2\right)_{n+1}(-1)^n q^{2n+1}}{\left(-q^2;q^2\right)_n} = \sum_{n\geq 1}\frac{\left(-q;q^2\right)_n q^{n^2}}{\left(-q^2;q^2\right)_n^2} - q. \tag{5.3}$$

Using Lemma 2.11 with $q \mapsto q^2, \zeta = c = -1$, and $d = \frac{1}{q}$, we have

$$\sum_{n\geq 1}\frac{\left(-q;q^2\right)_n q^{n^2}}{\left(-q^2;q^2\right)_n^2} = \frac{1+q}{1-q}\sum_{n\geq 1}\frac{\left(\frac{1}{q};q^2\right)_n(-1)^n q^{2n}}{\left(-q^2;q^2\right)_n}.$$

Thus, to prove (5.3), we need to show that

$$\frac{1+q}{1-q}\sum_{n\geq 1}\frac{\left(\frac{1}{q};q^2\right)_n(-1)^n q^{2n}}{\left(-q^2;q^2\right)_n} - \sum_{n\geq 1}\frac{\left(\frac{1}{q};q^2\right)_{n+1}(-1)^n q^{2n+1}}{\left(-q^2;q^2\right)_n} = q. \tag{5.4}$$

We prove this by the telescoping method. For $n \in \mathbb{N}$, define

$$\alpha_n := \alpha_n(q) := \frac{\left(\frac{1}{q};q^2\right)_n(-1)^n q^{2n}}{(1-q)\left(-q^2;q^2\right)_{n-1}}. \tag{5.5}$$

Note that

$$\alpha_n - \alpha_{n+1} = \frac{(1+q)\left(\frac{1}{q};q^2\right)_n(-1)^n q^{2n}}{(1-q)\left(-q^2;q^2\right)_n} - \frac{\left(\frac{1}{q};q^2\right)_{n+1}(-1)^n q^{2n+1}}{\left(-q^2;q^2\right)_n}.$$

Consequently, we have

$$\frac{1+q}{1-q}\sum_{n=1}^{N}\frac{\left(\frac{1}{q};q^2\right)_n(-1)^n q^{2n}}{\left(-q^2;q^2\right)_n} - \sum_{n=1}^{N}\frac{\left(\frac{1}{q};q^2\right)_{n+1}(-1)^n q^{2n+1}}{\left(-q^2;q^2\right)_n} = \sum_{n=1}^{N}(\alpha_n - \alpha_{n+1}) = \alpha_1 - \alpha_{N+1}.$$

From (5.5), we see that

$$\alpha_n = \frac{\left(\frac{1}{q};q^2\right)_n (-1)^n q^{2n}}{(1-q)\,(-q^2;q^2)_{n-1}} = \frac{(-1)^{n+1}\,(q;q^2)_{n-1}\,q^{2n-1}}{(-q^2;q^2)_{n-1}}.$$

Therefore, it follows that $\alpha_1 = q$ and $\alpha_N \to 0$ as $N \to \infty$. This concludes the proof of (5.4). $\square$

To prove Theorem 1.8, we use the following lemma.

Lemma 5.1. *We have*

$$\sum_{n\geq 1} \frac{q^{(2n-1)(n-1)}}{(1+q^{2n})\,(-q)_{2n-2}} = 1 - q\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}}.$$

Proof. Making the change of variable $n \mapsto n+1$, we have

$$\sum_{n\geq 1} \frac{q^{(2n-1)(n-1)}}{(1+q^{2n})\,(-q)_{2n-2}} = \frac{1}{1+q^2}\sum_{n\geq 0}\frac{q^{2n^2+n}}{(-q,-q^4;q^2)_n}. \tag{5.6}$$

Now, applying Lemma 2.7 with $q \mapsto q^2, a = -q^2, b = -q^{-1}$ and using Lemma 2.14 with $q \mapsto q^2, \alpha = 0$, $\beta = -q^4$, and $t = -q$, we have

$$\sum_{n\geq 0}\frac{q^{2n^2+n}}{(-q,-q^4;q^2)_n} = 1 + q^2 - q^2\sum_{n\geq 0}\frac{q^{2n^2+3n}}{(-q^4;q^2)_n\,(-q;q^2)_{n+1}}.$$

Applying this to (5.6) and then letting $n \mapsto n-1$, the lemma follows. $\square$

Proof of Theorem 1.8. From (1.11) and (5.1), we have

$$\sum_{n\geq 0}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+1}}{(\zeta q^{2n+2},q^{2n+3};q^2)_\infty} + \sum_{n\geq 1}\frac{\left(-q^{2n+1};q^2\right)_\infty q^{2n}}{(\zeta q^{2n},q^{2n+1};q^2)_\infty}$$
$$= \frac{(-q;q^2)_\infty}{\zeta q\,(\zeta q^2,q^3;q^2)_\infty}\left(\sum_{n\geq 1}\frac{\left(-\frac{1}{q};q^2\right)_n \zeta^n q^{n(n+1)}}{(-\zeta q,-q^2;q^2)_n} - \frac{(\zeta q^2,q^3;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n\geq 1}\frac{\zeta^n q^{2n^2-n}}{(-\zeta q,-q^2;q^2)_n}\right).$$

It is not hard to see that this is equivalent to

$$\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+1}}{(\zeta q^{2n+2},q^{2n+3};q^2)_\infty} + \sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+2}}{(\zeta q^{2n+2},q^{2n+3};q^2)_\infty} + \frac{1+q}{\zeta q}\sum_{n\geq 1}\frac{\zeta^n q^{2n^2-n}}{(-\zeta q,-q^2;q^2)_n}$$
$$= \frac{(-q;q^2)_\infty}{\zeta q\,(\zeta q^2,q^3;q^2)_\infty}\sum_{n\geq 1}\frac{\left(-\frac{1}{q};q^2\right)_n \zeta^n q^{n(n+1)}}{(-\zeta q,-q^2;q^2)_n} - \frac{(-q^3;q^2)_\infty\, q}{(\zeta q^2,q^3;q^2)_\infty} - \frac{(-q^3;q^2)_\infty\, q^2}{(\zeta q^2,q^3;q^2)_\infty}.$$

Taking the limit $\zeta \to q^{-2}$, we obtain

$$\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+1}}{(q^{2n},q^{2n+3};q^2)_\infty} + \sum_{n\geq 1}\frac{\left(-q^{2n+3},q^2\right)_\infty q^{2n+2}}{(q^{2n},q^{2n+3};q^2)_\infty} + q(1+q)\sum_{n\geq 1}\frac{q^{2n^2-3n}}{\left(-\frac{1}{q},-q^2;q^2\right)_n} \tag{5.7}$$
$$= \lim_{\zeta\to q^{-2}}\frac{\dfrac{(-q;q^2)_\infty}{\zeta q\,(\zeta q^4,q^3;q^2)_\infty}\displaystyle\sum_{n\geq 1}\frac{\left(-\frac{1}{q};q^2\right)_n \zeta^n q^{n(n+1)}}{(-\zeta q,-q^2;q^2)_n} - \dfrac{(-q;q^2)_\infty\, q}{(\zeta q^4,q^3;q^2)_\infty}}{1-\zeta q^2} =: L(q).$$

We define

$$N_r(\zeta;q) := \frac{\left(-q;q^2\right)_\infty}{\zeta q\left(\zeta q^4, q^3;q^2\right)_\infty} \sum_{n\geq 1} \frac{\left(-\frac{1}{q};q^2\right)_n \zeta^n q^{n(n+1)}}{\left(-\zeta q, -q^2;q^2\right)_n} - \frac{\left(-q;q^2\right)_\infty q}{\left(\zeta q^4, q^3;q^2\right)_\infty},\ N_r(q) := N_r\!\left(q^{-2};q\right).$$

Using Lemma 2.5, we see that $N_r(q) = 0$. Therefore, applying L'Hospital's rule, it follows that

$$\left[\frac{\partial}{\partial\zeta} N_r(\zeta;q)\right]_{\zeta=q^{-2}} = \frac{\left(-q;q^2\right)_\infty q}{\left(q^2\right)_\infty}\left(-q^2 + q^2\sum_{n\geq 1}\frac{nq^{n(n-1)}}{\left(-q^2;q^2\right)_n} - \sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(-q^2;q^2\right)_n}\sum_{k=1}^{n}\frac{q^{2k-1}}{1+q^{2k-3}}\right).$$

Plugging this into (5.7), we obtain

$$\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+1}}{\left(q^{2n}, q^{2n+3};q^2\right)_\infty} + \sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n+2}}{\left(q^{2n}, q^{2n+3};q^2\right)_\infty} + q(1+q)\sum_{n\geq 1}\frac{q^{2n^2-3n}}{\left(-\frac{1}{q}, -q^2;q^2\right)_n}$$
$$= -\frac{\left(-q;q^2\right)_\infty}{q\left(q^2\right)_\infty}\left(-q^2 + q^2\sum_{n\geq 1}\frac{nq^{n(n-1)}}{\left(-q^2;q^2\right)_n} - \sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(-q^2;q^2\right)_n}\sum_{k=1}^{n}\frac{q^{2k-1}}{1+q^{2k-3}}\right).$$

Simplifying the left-hand side and then dividing both sides by $q(1+q)$, we get

$$\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n}}{\left(q^{2n}, q^{2n+3};q^2\right)_\infty} + \sum_{n\geq 1}\frac{q^{2n^2-3n}}{\left(-\frac{1}{q}, -q^2;q^2\right)_n}$$
$$= \frac{\left(-q^3;q^2\right)_\infty}{\left(q^2\right)_\infty}\left(1 - \sum_{n\geq 1}\frac{nq^{n(n-1)}}{\left(-q^2;q^2\right)_n} + \sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(-q^2;q^2\right)_n}\sum_{k=1}^{n}\frac{q^{2k-3}}{1+q^{2k-3}}\right).$$

Letting $k \mapsto k+1$ and then using Lemma 2.6 with $q \mapsto q^2$, we obtain

$$\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n}}{\left(q^{2n}, q^{2n+3};q^2\right)_\infty} = \frac{\left(-q^3;q^2\right)_\infty}{\left(q^2\right)_\infty}\left(1 - \sigma\left(q^2\right) + \sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(-q^2;q^2\right)_n}\sum_{k=0}^{n-1}\frac{q^{2k-1}}{1+q^{2k-1}}\right)$$
$$- \sum_{n\geq 1}\frac{q^{2n^2-3n}}{\left(-\frac{1}{q}, -q^2;q^2\right)_n}. \tag{5.8}$$

We then simplify, using Lemma 5.1,

$$\sum_{n\geq 1}\frac{q^{2n^2-3n}}{\left(-\frac{1}{q}, -q^2;q^2\right)_n} = \frac{1}{1+q}\sum_{n\geq 1}\frac{q^{(2n-1)(n-1)}}{\left(1+q^{2n}\right)(-q)_{2n-2}} = \frac{1}{1+q}\left(1 - q\sum_{n\geq 1}\frac{q^{2n^2-n}}{(-q)_{2n}}\right)$$
$$= \frac{1}{1+q}\left(1 - q\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{\frac{n(3n-1)}{2}}\right),$$

where the last equality follows from (3.6). Plugging this into (5.8), we obtain

$$\sum_{n\geq 1}\frac{\left(-q^{2n+3};q^2\right)_\infty q^{2n}}{\left(q^{2n}, q^{2n+3};q^2\right)_\infty} = \frac{\left(-q^3;q^2\right)_\infty}{\left(q^2\right)_\infty}\left(1 - \sigma\left(q^2\right) + \sum_{n\geq 1}\frac{q^{n(n-1)}}{\left(-q^2;q^2\right)_n}\sum_{k=0}^{n-1}\frac{q^{2k-1}}{1+q^{2k-1}}\right)$$
$$- \frac{1}{1+q}\left(1 - q\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{\frac{n(3n-1)}{2}}\right). \tag{5.9}$$

Rearranging the double sum on the left-hand side of (5.9), we have

$$\sum_{n\geq 1}\frac{q^{n(n-1)}}{(-q^2;q^2)_n}\sum_{k=0}^{n-1}\frac{q^{2k-1}}{1+q^{2k-1}}=\sum_{k\geq 0}\frac{q^{k^2+3k-1}}{(1+q^{2k-1})(-q^2;q^2)_{k+1}}\lim_{t\to 0}\sum_{n\geq 0}\frac{\left(-\frac{q}{t};q^2\right)_n\left(tq^{2k+1}\right)^n}{(-q^{2k+4};q^2)_n}. \tag{5.10}$$

Applying Lemma 2.8 with $q\mapsto q^2$, $a=-\frac{q}{t}, b=q^2,$ and $c=-q^{2k+4}$, we have

$$\lim_{t\to 0}\sum_{n\geq 0}\frac{\left(-\frac{q}{t};q^2\right)_n\left(tq^{2k+1}\right)^n}{(-q^{2k+4};q^2)_n}=\lim_{t\to 0}\frac{\left(tq^{2k+3},-q^{2k+2};q^2\right)_\infty}{(-q^{2k+4},tq^{2k+1};q^2)_\infty}=1+q^{2k+2}.$$

Consequently, from (5.10), it follows that

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{n(n-1)}}{(-q^2;q^2)_n}\sum_{k=0}^{n-1}\frac{q^{2k-1}}{1+q^{2k-1}}&=\sum_{k\geq 0}\frac{q^{k^2+3k-1}}{(1+q^{2k-1})(-q^2;q^2)_k}=\frac{1}{1+q}+\sum_{k\geq 0}\frac{q^{k^2+5k+3}}{(1+q^{2k+1})(-q^2;q^2)_{k+1}}\\&=\frac{1}{1+q}+\sum_{k\geq 1}\frac{q^{k(k+1)}}{(-q^2;q^2)_k}-\sum_{k\geq 0}\frac{q^{k^2+3k+2}}{(1+q^{2k+1})(-q^2;q^2)_{k+1}}\\&=\frac{1}{1+q}+\sum_{k\geq 1}\frac{q^{k(k+1)}}{(-q^2;q^2)_k}-q\sum_{k\geq 0}\frac{q^{k(k+1)}}{(-q^2;q^2)_{k+1}}+\sum_{k\geq 0}\frac{q^{k(k+1)+1}}{(1+q^{2k+1})(-q^2;q^2)_{k+1}}\\&=\frac{1}{1+q}+\sigma\left(q^2\right)-1-q+\sum_{k\geq 0}\frac{q^{k(k+1)+1}}{(1+q^{2k+1})(-q^2;q^2)_{k+1}},\end{aligned}$$

using Lemma 2.5 and (1.5). Thus,

$$1-\sigma\left(q^2\right)+\sum_{n\geq 1}\frac{q^{n(n-1)}}{(-q^2;q^2)_n}\sum_{k=0}^{n-1}\frac{q^{2k-1}}{1+q^{2k-1}}=\frac{1}{1+q}-q+q\sum_{k\geq 0}\frac{q^{k(k+1)}}{(1+q^{2k+1})(-q^2;q^2)_{k+1}}. \tag{5.11}$$

Letting $k\mapsto k-1$, we have, using Lemma 2.5, (3.5), and (1.7),

$$\begin{aligned}\sum_{k\geq 0}\frac{q^{k(k+1)}}{(1+q^{2k+1})(-q^2;q^2)_{k+1}}&=\sum_{k\geq 0}\frac{q^{k(k+1)}}{(-q^2;q^2)_{k+1}}-q^{-1}\sum_{k\geq 1}\frac{q^{k(k+1)}}{(1+q^{2k-1})(-q^2;q^2)_k}\\&=1-q^{-1}\sum_{k\geq 1}\frac{q^{k(k+1)}}{(1+q^{2k-1})(-q^2;q^2)_k}=\frac{1}{1-q}\left(W_1(q)-q\right).\end{aligned}$$

Plugging this into (5.11), we obtain

$$1-\sigma\left(q^2\right)+\sum_{n\geq 1}\frac{q^{n(n-1)}}{(-q^2;q^2)_n}\sum_{k=0}^{n-1}\frac{q^{2k-1}}{1+q^{2k-1}}=\frac{q}{1-q}W_1(q)+\frac{1-2q-q^2}{1-q^2}.$$

Plugging this into (5.9) and using (1.15), we conclude the proof. $\square$

Proof of Theorem 1.9. From (1.16), we have

$$\overline{P}_{\mathrm{ev}}(\zeta;q)=\frac{\zeta q\,(1+q)\left(-q^2,-\zeta q^3;q^2\right)_\infty}{(\zeta q^2;q^2)_\infty}\sum_{n\geq 0}\frac{\left(\zeta q^2;q^2\right)_n q^{2n}}{(-q^2,-\zeta q^3;q^2)_n}. \tag{5.12}$$

Applying Lemma 2.2 with $q\mapsto q^2, a=1, b=\zeta q,$ and $c=\zeta$, we have

$$\sum_{n\geq 0}\frac{\left(\zeta q^2;q^2\right)_n q^{2n}}{(-q^2,-\zeta q^3;q^2)_n}=-\frac{2\,(1+\zeta q)}{\zeta q}\sum_{n\geq 1}\frac{\left(\zeta q^2;q^2\right)_{n-1}(-1)^n\zeta^n q^{n^2}}{(-\zeta q,-\zeta q^2;q^2)_n}$$

$$+\frac{\left(\zeta q^2;q^2\right)_\infty}{\zeta q\left(-q^2,-\zeta q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{(-1)^n\zeta^n q^{n^2}}{\left(-\zeta q^2;q^2\right)_n}. \tag{5.13}$$

Applying Lemma 2.10 with $q\mapsto q^2, b=-\zeta q, c=-\zeta$, and $d=-1$, it follows that

$$\sum_{n\geq 1}\frac{\left(-\frac{\zeta q}{a},\zeta;q^2\right)_n(-a)^n}{\left(-\zeta q,-\zeta q^2;q^2\right)_n}=\frac{(a+\zeta q)\left(-1+\zeta\right)}{-a+\zeta q}\sum_{n\geq 0}\frac{\left(a,-q;q^2\right)_n(-\zeta)^n}{\left(-\zeta q,-a;q^2\right)_n}\left(\frac{-aq^{2n}}{1+aq^{2n}}+\frac{\zeta q^{2n+1}}{1+\zeta q^{2n+1}}\right). \tag{5.14}$$

Letting $a\to 0$ on the left-hand side of (5.14), we have, using $\lim_{a\to 0}(-\frac{\zeta q}{a};q^2)_n a^n=\zeta^n q^{n^2}$,

$$\lim_{a\to 0}\sum_{n\geq 1}\frac{\left(-\frac{\zeta q}{a},\zeta;q^2\right)_n(-a)^n}{\left(-\zeta q,-\zeta q^2;q^2\right)_n}=(1-\zeta)\sum_{n\geq 1}\frac{\left(\zeta q^2;q^2\right)_{n-1}(-1)^n\zeta^n q^{n^2}}{\left(-\zeta q,-\zeta q^2;q^2\right)_n}. \tag{5.15}$$

Letting $a\to 0$ on the right-hand side of (5.14) and substituting the resulting identity and (5.15) into (5.14) and then dividing by $1-\zeta$, we get

$$\sum_{n\geq 1}\frac{\left(\zeta q^2;q^2\right)_{n-1}(-1)^n\zeta^n q^{n^2}}{\left(-\zeta q^2,-\zeta q;q^2\right)_n}=-\frac{\zeta q}{1+\zeta q}\sum_{n\geq 0}\frac{\left(-q;q^2\right)_n(-1)^n\zeta^n q^{2n}}{\left(-\zeta q^3;q^2\right)_n}.$$

Plugging this into (5.13), we then obtain

$$\begin{aligned}\sum_{n\geq 0}\frac{\left(\zeta q^2;q^2\right)_n q^{2n}}{\left(-q^2,-\zeta q^3;q^2\right)_n}&=2\sum_{n\geq 0}\frac{\left(-q;q^2\right)_n(-1)^n\zeta^n q^{2n}}{\left(-\zeta q^3;q^2\right)_n}+\frac{\left(\zeta q^2;q^2\right)_\infty}{\zeta q\left(-q^2,-\zeta q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{(-1)^n\zeta^n q^{n^2}}{\left(-\zeta q^2;q^2\right)_n}\\
&=\frac{\left(q,\zeta q^2\right)_\infty\left(\frac{1}{\zeta q};q^2\right)_\infty}{\left(-\zeta q^2\right)_\infty}+\left(1+\frac{1}{\zeta q}\right)\sum_{n\geq 0}\frac{\left(-\frac{1}{\zeta};q^2\right)_n(-1)^n q^n}{\left(-q^2;q^2\right)_n}+\frac{\left(\zeta q^2;q^2\right)_\infty}{\zeta q\left(-q^2,-\zeta q^3;q^2\right)_\infty}\sum_{n\geq 1}\frac{(-1)^n\zeta^n q^{n^2}}{\left(-\zeta q^2;q^2\right)_n}.\end{aligned}$$

using Lemma 2.13. Consequently, by (5.12), we get

$$\begin{aligned}\overline{P}_{\mathrm{ev}}(\zeta;q)=\frac{\zeta q\left(q^2,\zeta q^3,\frac{1}{\zeta q};q^2\right)_\infty}{\left(-q^3,-\zeta q^2;q^2\right)_\infty}+\frac{(1+q)\left(-q^2,-\zeta q;q^2\right)_\infty}{\left(\zeta q^2;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(-\frac{1}{\zeta};q^2\right)_n(-1)^n q^n}{\left(-q^2;q^2\right)_n}&\\
+(1+q)\sum_{n\geq 1}\frac{(-1)^n\zeta^n q^{n^2}}{\left(-\zeta q^2;q^2\right)_n}.&\end{aligned} \tag{5.16}$$

Using the definition of ν from (1.17), we conclude the proof. □

Proof of Corollary 1.10. Plugging $\zeta=-q$ into Theorem 1.9, (1.17), and (1.18), we have

$$\begin{aligned}\overline{P}_{\mathrm{ev}}(-q;q)=-(1+q)-\frac{2\left(-q^2;q^4\right)_\infty\left(q^8;q^8\right)_\infty}{\left(q^6;q^4\right)_\infty}+\frac{(1+q)\left(q^4;q^4\right)_\infty}{\left(-q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(\frac{1}{q};q^2\right)_n(-1)^n q^n}{\left(-q^2;q^2\right)_n}&\\
+\left(1-q^2\right)\nu(-q).&\end{aligned} \tag{5.17}$$

Using Lemma 2.14 with $q\mapsto q^2, \alpha=\frac{1}{q}, \beta=-q^2$, and $t=-q$, we obtain

$$\sum_{n\geq 0}\frac{\left(\frac{1}{q};q^2\right)_n(-1)^n q^n}{\left(-q^2;q^2\right)_n}=\frac{2}{1+q},$$

employing the fact $(1;q^2)_n=0$ for $n\in\mathbb{N}$. Plugging this into (5.17), we conclude the proof. □

Proof of Corollary 1.11. Using (1.17), we obtain

$$(1+\zeta)\,\nu\left(iq, i\,(\zeta q)^{\frac{1}{2}}\,; q\right) = \sum_{n\geq 0} \frac{(-1)^n \zeta^n q^{n^2}}{\left(-\zeta q^2; q^2\right)_n}.$$

Plugging this into Theorem 1.9 with $\zeta = -1$, we obtain

$$\overline{P}_{\mathrm{ev}}(-1;q) = -\,(1+q)\left(-q;q^2\right)_\infty + (1+q)\left(q;q^2\right)_\infty + (1+q)\left(\sum_{n\geq 0}\frac{q^{n^2}}{\left(q^2;q^2\right)_n} - 1\right).$$

Applying Lemma 2.3 to this, we conclude the proof. $\square$

Proof of Corollary 1.12. Using (1.16), splitting off the terms $n = 1$ on both of the sums on the left-hand side of (5.16), rearranging, and letting $\zeta \to q^{-2}$, we obtain

$$\sum_{n\geq 1}\frac{\left(-q^{2n+2}, -q^{2n+1}; q^2\right)_\infty q^{2n-1}}{\left(q^{2n};q^2\right)_\infty} + \sum_{n\geq 1}\frac{\left(-q^{2n+2}, -q^{2n+1}; q^2\right)_\infty q^{2n}}{\left(q^{2n};q^2\right)_\infty} - \frac{\left(q^2, q, q; q^2\right)_\infty}{q\left(-q^3, -1; q^2\right)_\infty}$$
$$- \,(1+q)\sum_{n\geq 1}\frac{(-1)^n q^{n^2-2n}}{\left(-1;q^2\right)_n} = \frac{\mathcal{L}(q)}{\left(q^2;q^2\right)_\infty}, \tag{5.18}$$

where

$$\mathcal{L}(q) := \lim_{\zeta\to q^{-2}} \frac{(1+q)\left(-q^2, -\zeta q; q^2\right)_\infty \sum_{n\geq 0}\frac{\left(-\frac{1}{\zeta};q^2\right)_n (-1)^n q^n}{\left(-q^2;q^2\right)_n} - \left(-q^2, -\zeta q^3; q^2\right)_\infty \zeta q\,(1+q)}{1-\zeta q^2}. \tag{5.19}$$

Applying L'Hospital, Lemma 2.12, (1.4), and (5.19), we obtain

$$\frac{\mathcal{L}(q)}{\left(q^2;q^2\right)_\infty} = \frac{(1{+}q)\left(1-\Theta^2(-q)\right)}{4q\,(q)_\infty}.$$

Plugging this into (5.18), using (1.9), using (1.4), [9, equation (1.6)], [27, equation (26.27)], and (1.13), we conclude the proof. $\square$

Proof of Theorem 1.14. By Lemma 2.1 with $q \mapsto q^2$, $a = -\zeta q^{-1}, b = -\zeta^{-1}q^{-1}$, and $c = q^{-1}$,

$$\sum_{n\geq 0}\frac{\left(\zeta q, \zeta^{-1}q; q^2\right)_n q^{2n+2}}{\left(-q;q^2\right)_n} = \sum_{n\geq 1}\frac{\left(-q;q^2\right)_n q^{n^2+1}}{\left(-\zeta q^2, -\zeta^{-1}q^2; q^2\right)_n} - \frac{\left(\zeta q, \zeta^{-1}q; q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\sum_{n\geq 1}\frac{q^{2n^2+1}}{\left(-\zeta q^2, -\zeta^{-1}q^2; q^2\right)_n}.$$

Applying (1.2) and (1.3), we get

$$\sum_{n\geq 0}\frac{\left(\zeta q, \zeta^{-1}q; q^2\right)_n q^{2n}}{\left(-q;q^2\right)_n} = q^{-1}\left(\mathrm{R2}\left(-\zeta;q\right) - 1 - \frac{\left(\zeta q, \zeta^{-1}q; q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\left(R\left(-\zeta;q^2\right) - 1\right)\right).$$

Using this and (1.19), we then conclude the proof. $\square$

6. Concluding Remarks

We propose the following two questions for future research.

(1) In Theorems 1.6 and 1.9, for special choices of ζ, the functions $P_{\mathrm{od}}(\zeta;q)$ and $P_{\mathrm{ev}}(\zeta;q)$ are connected to various modular objects. Do $f(\zeta q, q^2; q^2)$ and $\nu(iq, i(\zeta q)^{\frac{1}{2}}; q)$ fit into the theory

of mock modular forms if ζ is a root of unity? In Theorem 1.6 and Theorem 1.9, for specific choices of ζ, we study the modularity of

$$F_1(\zeta;q) := \sum_{n\geq 1} \frac{\left(-\frac{1}{q};q^2\right)_n \zeta^n q^{n(n+1)}}{(-\zeta q, -q^2;q^2)_n} \quad \text{and} \quad F_2(\zeta;q) := \sum_{n\geq 0} \frac{\left(-\frac{1}{\zeta};q^2\right)_n (-1)^n q^n}{(-q^2;q^2)_n}.$$

Are $F_1(\zeta;q)$ or $F_2(\zeta;q)$ modular for other choices?

(2) Give a bijective proof of (1.10).

References

[1] R. Agarwal, *A family of basic hypergeometric and combinatorial identities and certain summation formulae*, Indian J. pure appl. Math. **12** (1981), 728–737.

[2] G. Andrews, *Ramanujan's "lost" notebook I. Partial θ-functions*, Adv. Math. **41** (1981), 137–172.

[3] G. Andrews, *Ramanujan's "lost" notebook IV: stacks and alternating parity in partitions*, Adv. Math. **53** (1984), 55–74.

[4] G. Andrews, *Ramanujan's "lost" notebook V: Euler's partition identity*, Adv. Math. **61** (1986), 156–164.

[5] G. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998.

[6] G. Andrews and B. Berndt, *Ramanujan's lost notebook*, Part II, Springer, New York, 2010.

[7] G. Andrews and B. Berndt, *Ramanujan's lost notebook*, Part V, Springer, New York, 2018.

[8] G. Andrews, F. Dyson, and D. Hickerson, *Partitions and indefinite quadratic forms*, Invent. Math. **91** (1988), 391–407.

[9] G. Andrews and R. Kumar, *Rank, two-color partitions and mock theta functions*, Proc. Amer. Math. Soc. **153** (2025), 4669–4682.

[10] G. Andrews, R. Rhoades, and S. Zwegers, *Modularity of the concave composition generating function*, Algebra Number Theory **7** (2013), 2103–2139.

[11] A. Atkin and P. Swinnerton-Dyer, *Some properties of partitions*, Proc. Lond. Math. Soc. (3) **3** (1954), 84–106.

[12] A. Berkovich and F. Garvan, *Some observations on Dyson's new symmetries of partitions*, J. Combin. Theory Ser. A **100** (2002), 61–93.

[13] B. Berndt, *Number theory in the spirit of Ramanujan*, Student Mathematical Library, 34, Amer. Math. Soc., Providence, RI, 2006.

[14] B. Berndt and R. Rankin, *Ramanujan: letters and commentary*, History of Mathematics **9** (1995), Amer. Math. Soc., Providence, RI.

[15] S. Bhoria, P. Eyyunni, and B. Maji, *Generalizations of five q-series identities of Ramanujan and unexplored weighted partition identities*, Ramanujan J. **58** (2022), 435–462.

[16] K. Bringmann, W. Craig, and C. Nazaroglu, *On the asymptotic behavior for partitions separated by parity*, Electron. J. Combin. **32:1** (2025), 1–18.

[17] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. Math. **171** (2010), 419–449.

[18] K. Bringmann, K. Ono, and R. Rhoades, *Eulerian series as modular forms*, J. Amer. Math. Soc. **21** (2008), 1085–1104.

[19] K. Bringmann, C. Jennings-Shaffer, and K. Mahlburg, *On a Tauberian theorem of Ingham and Euler–Maclaurin summation*, Ramanujan J. **61** (2021), 55–86.

[20] K. Bringmann and C. Nazaroglu, *A framework for modular properties of false theta functions*, Res. Math. Sci. **6:30** (2019).

[21] Y. Choi, *The basic bilateral hypergeometric series and the mock theta functions*, Ramanujan J. **24** (2011), 345–386.

[22] H. Cohen, *q-identities for Maass waveforms*, Invent. Math. **91** (1988), 409–422.

[23] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), 1623–1635.

[24] D. Corson, D. Favero, K. Liesinger, and S. Zubairy, *Characters and q-series in* $\mathbb{Q}(\sqrt{2})$, J. Number Theory **107** (2004), 392–405.

[25] J. Donato, *On minimal excludant over overpartitions*, arXiv:2507.04402v1.

[26] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), 10–15.

[27] N. Fine, *Basic hypergeometric series and applications*, Amer. Math. Soc. Providence, RI, 1988.

[28] F. Garvan, *Weighted partition identities and divisor sums*, Frontiers in orthogonal polynomials and q-Series. (2018), 239–249.

[29] G. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. Lond. Math. Soc. (3) **17** (1918), 75–115.

[30] A. Ingham, *A Tauberian theorem for partitions*, Ann. Math. **42** (1941), 1075–1090.
[31] S. Kang, *Generalizations of Ramanujan's reciprocity theorem and their applications*, J. Lond. Math. Soc. (2) **75** (2007), 18–34.
[32] J. Lovejoy and R. Osburn, *M_2-rank differences for partitions without repeated odd parts*, J. Théor. Nombres Bordeaux **21** (2009), 313–334.
[33] R. McIntosh, *Second order mock theta functions*, Canad. Math. Bull. **50** (2007), 284–290.
[34] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Math. Proc. Cambridge Philos. Soc. **XIX** (1919), 207–210.
[35] G. Watson, *The final problem: An account of the mock-theta functions*, J. Lond. Math. Soc. (2) **11** (1936), 55–80.
[36] D. Zagier, *The Mellin transform and related analytic techniques*, Appendix to E. Zeidler, Quantum field theory I: Basics in mathematics and physics. A bridge between mathematicians and physicists, 305–323, Springer, Berlin (2006).
[37] D. Zagier, *Quantum modular forms*, In: Quanta of Math: Conference in honor of Alain Connes, Clay Mathematics Proceedings 11, AMS and Clay Mathematics Institute, (2010), 656–675.
[38] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universität Utrecht (2002).
[39] S. Zwegers, *Mock Maass theta functions*, Q. J. Math. **63** (2012), 753–770,

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