A CLASS OF IDENTITIES ASSOCIATED WITH DIRICHLET SERIES SATISFYING HECKE'S FUNCTIONAL EQUATION

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ABSTRACT. We consider two sequences $a(n)$ and $b(n)$, $1 \leq n < \infty$, generated by Dirichlet series of the forms

$$\sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s},$$

satisfying a familiar functional equation involving the gamma function $\Gamma(s)$. A general identity is established. Appearing on one side is an infinite series involving $a(n)$ and modified Bessel functions K_ν, wherein on the other side is an infinite series involving $b(n)$ that is an analogue of the Hurwitz zeta function. Seven special cases, including $a(n) = \tau(n)$ and $a(n) = r_k(n)$, are examined, where $\tau(n)$ is Ramanujan's arithmetical function and $r_k(n)$ denotes the number of representations of n as a sum of k squares. Most of the six special cases appear to be new.

1. INTRODUCTION

Our goal is to establish a new set of identities involving arithmetical functions whose generating functions are Dirichlet series satisfying Hecke's functional equation. Our general theorem involves the modified Bessel function $K_\nu(z)$. Even in the special case [16, pp. 79, 80, no. (13)]

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \tag{1.1}$$

most special cases are new.

We employ the setting of K. Chandrasekharan and R. Narasimhan from their paper [7]. Throughout our paper, $\sigma = \text{Re}(s)$. Let $a(n)$ and $b(n)$, $1 \leq n < \infty$, be two sequences of complex numbers, not identically 0. Set

$$\varphi(s) := \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s}, \quad \sigma > \sigma_a; \qquad \psi(s) := \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s}, \quad \sigma > \sigma_a^*, \tag{1.2}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences of positive numbers, each tending to ∞, and σ_a and σ_a^* are the (finite) abscissae of absolute convergence for $\varphi(s)$ and $\psi(s)$, respectively. We assume that $\varphi(s)$ and $\psi(s)$ have analytic continuations into the entire complex plane $\mathbb{C}$ and are analytic on $\mathbb{C}$ except for a finite set S of poles. Suppose that for some $\delta > 0$, $\varphi(s)$ and $\psi(s)$ satisfy a functional equation of the form

$$\chi(s) := (2\pi)^{-s}\Gamma(s)\varphi(s) = (2\pi)^{s-\delta}\Gamma(\delta - s)\psi(\delta - s). \tag{1.3}$$

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Chandrasekharan and Narasimhan show that the functional equation (1.3) is equivalent to the following two identities [7, p. 6, Lemmas 4, 5] the first of which is due to Bochner [6].

Theorem 1.1. *The functional equation* (1.2) *is equivalent to the 'modular' relation*

$$\sum_{n=1}^{\infty} a(n)e^{-\lambda_n x} = \left(\frac{2\pi}{x}\right)^{\delta} \sum_{n=1}^{\infty} b(n)e^{-4\pi^2\mu_n/x} + P(x), \qquad \text{Re}(x) > 0,$$

where

$$P(x) := \frac{1}{2\pi i}\int_{\mathcal{C}} (2\pi)^z \chi(z) x^{-z} dz,$$

where $\mathcal{C}$ is a curve or curves encircling all of $\mathbf{S}$.

Theorem 1.2. *Let $J_\nu(z)$ denote the ordinary Bessel function of order ν. Let $x > 0$ and $\rho > 2\sigma_a^* - \delta - \frac{1}{2}$. Then the functional equation* (1.2) *is equivalent to the identity*

$$\frac{1}{\Gamma(\rho+1)} {\sum_{\lambda_n \leq x}}' a(n)(x-\lambda_n)^{\rho} = \left(\frac{1}{2\pi}\right)^{\rho} \sum_{n=1}^{\infty} b(n) \left(\frac{x}{\mu_n}\right)^{(\delta+\rho)/2} J_{\delta+\rho}(4\pi\sqrt{\mu_n x}) + Q_\rho(x), \tag{1.4}$$

where the prime $'$ on the summation sign on the left side indicates that if $\rho = 0$ and $x \in \{\lambda_n\}$, then only $\frac{1}{2}a(x)$ is counted. Furthermore, $Q_\rho(x)$ is defined by

$$Q_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\chi(z)(2\pi)^z x^{z+\rho}}{\Gamma(\rho+1+z)} dz, \tag{1.5}$$

where $\mathcal{C}$ is a curve or curves encircling $\mathbf{S}$.

The restriction $\rho > 2\sigma_a^* - \delta - \frac{1}{2}$ can be replaced by $\rho > 2\sigma_a^* - \delta - \frac{3}{2}$ under certain conditions given in [7, p. 14, Theorem III]. Because we later use analytic continuation, this extension is not important for us here.

Theorem 1.1 is not explicitly used in the sequel. However, Theorem 1.2 is the key to our main theorem, Theorem 3.1.

We conclude our paper with seven examples, including the following arithmetical functions: $r_k(n)$, the number of representations of n as a sum of k squares; $\sigma_k(n)$, the sum of the k powers of the divisors of n; Ramanujan's arithmetical function $\tau(n)$; $\chi(n)$, a primitive character; and $F(n)$, the number of integral ideals of norm n in an imaginary quadratic number field. The identity involving $r_k(n)$ is known (but not well known).

2. Preliminaries

We refer readers to G. N. Watson's classical treatise for the definitions of the Bessel functions $J_\nu(z)$ and $K_\nu(z)$ [16, pp. 15, 78]. The following lemmas will be used in the sequel.

Lemma 2.1. [16, pp. 199, 202] *Let $J_\nu(x)$ denote the ordinary Bessel function of order ν, and let $K_\nu(x)$ denote the modified Bessel function of order ν. As $z \to \infty$,*

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left\{ \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + O\left(\frac{1}{z}\right) \right\}, \tag{2.1}$$

$$K_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-z}\left\{1 + O\left(\frac{1}{z}\right)\right\}. \tag{2.2}$$

Lemma 2.2. [16, p. 79] *For each non-negative integer m and arbitrary ν,*

$$\left(\frac{d}{zdz}\right)^m \{z^\nu K_\nu(z)\} = (-1)^m z^{\nu-m} K_{\nu-m}(z). \tag{2.3}$$

Lemma 2.3. [5, p. 329] *For* $\mathrm{Re}(\nu) > 0$*,*

$$\lim_{z\to 0} z^\nu K_\nu(z) = 2^{\nu-1}\Gamma(\nu).$$

Lemma 2.4. [16, Equation (2), p. 410] *Assume that* $\mathrm{Re}(\nu) + 1 > |\mathrm{Re}(\mu)|$ *and that* $a, b > 0$*. Then*

$$\int_0^\infty t^{\mu+\nu+1} K_\mu(at) J_\nu(bt)dt = \frac{(2a)^\mu (2b)^\nu \Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}. \tag{2.4}$$

Lemma 2.5. [12, p. 708, no. 16] *For* $\mathrm{Re}(\mu + 1 \pm \nu) > 0$ *and* $\mathrm{Re}(a) > 0$*,*

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right).$$

3. The Primary Theorem

Theorem 3.1. *Assume that* $\mathrm{Re}(\nu) > 0$ *and* $\mathrm{Re}(s) > 0$*. Also assume that* $\delta + \rho + \nu + 1 > \sigma_a^* > 0$*. Assume that the integral on the right side below converges absolutely. Then,*

$$\begin{aligned}\frac{1}{\Gamma(\rho+1)} \sum_{n=1}^\infty a(n) \int_{\lambda_n}^\infty (x-\lambda_n)^\rho x^{\nu/2} K_\nu(s\sqrt{x})dx \\ = 2^{3\delta+2\rho+\nu+1} s^\nu \pi^\delta \Gamma(\delta+\rho+\nu+1) \sum_{n=1}^\infty \frac{b(n)}{(s^2+16\pi^2\mu_n)^{\delta+\rho+\nu+1}} \\ + \int_0^\infty Q_\rho(x) x^{\nu/2} K_\nu(s\sqrt{x})dx.\end{aligned} \tag{3.1}$$

Proof. Assume that $\rho > 2\sigma_a^* - \delta - \frac{1}{2}$. Multiply both sides of (1.4) by $x^{\nu/2}K_\nu(s\sqrt{x})$, where $s > 0$, and integrate over $0 \le x < \infty$. Let $F_1(\delta, \rho, \nu)$ denote the left-hand side and let $F_2(\delta, \rho, \nu)$ and $F_3(\delta, \rho, \nu)$ denote, in order, the two terms on the right-hand side that we so obtain.

First, on the left-hand side of (1.4), we readily find that, for $\mathrm{Re}(\nu) > 0$,

$$F_1(\delta, \rho, \nu) = \frac{1}{\Gamma(\rho+1)} \sum_{n=1}^\infty a(n) \int_{\lambda_n}^\infty (x-\lambda_n)^\rho x^{\nu/2} K_\nu(s\sqrt{x})dx. \tag{3.2}$$

Second, using Lemma 2.4, assuming that $\mathrm{Re}(\delta + \rho) + 1 > \mathrm{Re}(\nu)$, using Lemma 2.1, and inverting the order of summation and integration by absolute convergence, we have

$$F_2(\delta, \rho, \nu) = (2\pi)^{-\rho} \sum_{n=1}^\infty b(n)\mu_n^{-(\delta+\rho)/2} I(\delta, \rho, \nu), \tag{3.3}$$

where

$$\begin{aligned}I(\delta,\rho,\nu):=&\int_0^\infty x^{(\delta+\rho+\nu)/2}J_{\delta+\rho}(4\pi\sqrt{\mu_n x})K_\nu(s\sqrt{x})dx\\=&2\int_0^\infty t^{\delta+\rho+\nu+1}J_{\delta+\rho}(4\pi\sqrt{\mu_n}\,t)K_\nu(st)dt\\=&2\frac{(2s)^\nu(8\pi\sqrt{\mu_n})^{\delta+\rho}\Gamma(\delta+\rho+\nu+1)}{(s^2+16\pi^2\mu_n)^{\delta+\rho+\nu+1}}.\end{aligned}\tag{3.4}$$

Hence, with the use of (3.4) on the right side of (1.4), after simplification, we obtain the sum

$$F_2(\delta,\rho,\nu)=2^{3\delta+2\rho+\nu+1}s^\nu\pi^\delta\Gamma(\delta+\rho+\nu+1)\sum_{n=1}^\infty\frac{b(n)}{(s^2+16\pi^2\mu_n)^{\delta+\rho+\nu+1}}.\tag{3.5}$$

Now use analytic continuation in ρ,ν, and s to conclude that (3.5) is valid provided that $\delta+\rho+\nu+1>\sigma_a^*$ and $\mathrm{Re}(s)>0$.

Lastly, for the remaining term on the right side of (1.4), we find that

$$F_3(\delta,\rho,\nu)=\int_0^\infty Q_\rho(x)x^{\nu/2}K_\nu(s\sqrt{x})dx,\tag{3.6}$$

provided that $\mathrm{Re}(\nu)>0$ and $\mathrm{Re}(s)>0$, and that the integral above converges absolutely.

Bringing (3.2), (3.5), and (3.6) together, we complete the proof of Theorem 3.1.

□

Theorem 3.1 is a generalization of theorems proved by the first author [1], [2], [3, p. 154, Equation (6.11)]. Recalling the definition of the Hurwitz zeta function, observe that the series on the right-hand side of (3.1) is an analogue of the Hurwitz zeta function. Thus, (3.1) provides an analytic continuation for the series on the right side of (3.1).

4. Special Cases: Non-negative integer values of ρ

If we set $\rho=0$ in Theorem 3.1 and employ Lemmas 2.2 and 2.1, we find that the left-hand side of (3.1) is given by

$$\begin{aligned}\sum_{n=1}^\infty a(n)\int_{\lambda_n}^\infty x^{\nu/2}K_\nu(s\sqrt{x})dx=&\frac{2}{s^{\nu+2}}\sum_{n=1}^\infty a(n)\int_{s\sqrt{\lambda_n}}^\infty t^{\nu+1}K_\nu(t)dt\\=&-\frac{2}{s^{\nu+2}}\sum_{n=1}^\infty a(n)\int_{s\sqrt{\lambda_n}}^\infty\frac{d}{dt}\left\{t^{\nu+1}K_{\nu+1}(t)\right\}dt\\=&\frac{2}{s^{\nu+2}}\sum_{n=1}^\infty a(n)(s\sqrt{\lambda_n})^{\nu+1}K_{\nu+1}(s\sqrt{\lambda_n})\\=&\frac{2}{s}\sum_{n=1}^\infty a(n)\lambda_n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{\lambda_n}).\end{aligned}\tag{4.1}$$

Hence, we have established the following theorem.

Theorem 4.1. *Assume that* $\mathrm{Re}(\nu) > 0$ *and* $\mathrm{Re}(s) > 0$. *Also assume that* $\delta+\nu+1 > \sigma_a^* > 0$. *Assume that the integral on the right side below converges absolutely. Then,*

$$\frac{2}{s}\sum_{n=1}^{\infty} a(n)\lambda_n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{\lambda_n}) = 2^{3\delta+\nu+1}s^{\nu}\pi^{\delta}\Gamma(\delta+\nu+1)\sum_{n=1}^{\infty}\frac{b(n)}{(s^2+16\pi^2\mu_n)^{\delta+\nu+1}} + \int_0^{\infty} Q_0(x)x^{\nu/2}K_{\nu}(s\sqrt{x})dx. \tag{4.2}$$

Return to Theorem 3.1 and, for any $m \in \mathbb{N}$, let $\rho = m$. Apply the binomial theorem on the left-hand side. In the integrand we obtain polynomials in x of degree k, $0 \le k \le m$. For the integral involving x^k, integrate by parts k times with the aid of Lemma 2.2. With the help of the binomial theorem once again, simplify the double sum that arises. For the integral on the right-hand side of (3.1), integrate by parts m times with the help of Lemma 2.2. In conclusion, after simplification, we obtain Theorem 4.1 with ν replaced by $\nu+m$.

Chandrasekharan and Narasimhan [7, p. 8, Lemma 6] proved the following theorem, which is similar in appearance to Theorem 3.1 in the special case $\nu = 1/2$.

Theorem 4.2. *Let* ρ *denote a non-negative integer,* $\mathrm{Re}(s) > 0$, *and* $\rho > 2\sigma_a^* - \delta - \frac{1}{2}$. *Then*

$$\left(-\frac{1}{s}\frac{d}{ds}\right)^{\rho}\left\{\frac{1}{s}\sum_{n=1}^{\infty} a(n)e^{-s\sqrt{\lambda_n}}\right\}$$
$$=2^{3\delta+\rho}\Gamma(\delta+\rho+\tfrac{1}{2})\pi^{\delta-1/2}\sum_{n=1}^{\infty}\frac{b(n)}{(s^2+16\pi^2\mu_n)^{\delta+\rho+1/2}} + R_{\rho}(s),$$

where

$$R_{\rho}(s) := \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\chi(z)(2\pi)^z\Gamma(2z+2\rho+1)2^{-\rho}s^{-2z-2\rho-1}}{\Gamma(z+\rho+1)}dz.$$

5. EXAMPLE 1: $r_k(n)$

In the examples below we refer to calculations made by Chandrasekharan and Narasimhan [7] to illustrate Theorem 1.2. In particular, we use a few of their determinations of $Q_{\rho}(x)$.

Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares, where $k \ge 2$. Then

$$\zeta_k(s) := \sum_{n=1}^{\infty}\frac{r_k(n)}{n^s}, \qquad \sigma > k/2,$$

satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_k(s) = \pi^{s-k/2}\Gamma(k/2-s)\zeta_k(k/2-s). \tag{5.1}$$

In the notation of (1.3),

$$a(n) = b(n) = r_k(n), \qquad \delta = k/2, \qquad \text{and} \qquad \lambda_n = \mu_n = n/2.$$

From the functional equation (5.1), $\zeta_k(0) = -1$, and $\zeta_k(s)$ has a simple pole at $s = 2k$ with residue $\pi^{k/2}/\Gamma(k/2)$. We now apply Theorem 4.1. First, from the preceding remarks,

$$Q_0(x) = -1 + \frac{(2\pi)^{k/2}x^{k/2}}{\Gamma(1+k/2)}. \tag{5.2}$$

Second, we calculate the integral on the right side of (4.2). To that end,

$$\begin{aligned} I :=& \int_0^\infty \left(-1 + \frac{(2\pi)^{k/2}x^{k/2}}{\Gamma(1+k/2)}\right) x^{\nu/2} K_\nu(s\sqrt{x})dx \\ =& 2\int_0^\infty \left(-1 + \frac{(2\pi)^{k/2}t^k}{\Gamma(1+k/2)}\right) t^{\nu+1} K_\nu(st)dt \\ =:& I_1 + I_2. \end{aligned} \tag{5.3}$$

Using Lemmas 2.2, 2.1, and 2.3 in order, we find that

$$\begin{aligned} I_1 &= -2\int_0^\infty t^{\nu+1}K_\nu(st)dt \\ &= -\frac{2}{s^{\nu+2}}\int_0^\infty x^{\nu+1}K_\nu(x)dx \\ &= \frac{2}{s^{\nu+2}}\int_0^\infty \frac{d}{dx}\left\{x^{\nu+1}K_{\nu+1}(x)\right\}dx \\ &= -\frac{2^{\nu+1}}{s^{\nu+2}}\Gamma(\nu+1). \end{aligned} \tag{5.4}$$

Secondly, using Lemma 2.5, we find that

$$\begin{aligned} I_2 =& \frac{2(2\pi)^{k/2}}{\Gamma(1+k/2)}\int_0^\infty t^{k+\nu+1}K_\nu(st)dt \\ =& \frac{2(2\pi)^{k/2}}{s^{k+\nu+2}\Gamma(1+k/2)}\int_0^\infty x^{k+\nu+1}K_\nu(x)dx \\ =& \frac{2^{3k/2+\nu+1}\pi^{k/2}}{s^{k+\nu+2}}\Gamma(\nu+1+k/2). \end{aligned} \tag{5.5}$$

Putting (5.4) and (5.5) in (5.3), we conclude that

$$I = -\frac{2^{\nu+1}}{s^{\nu+2}}\Gamma(\nu+1) + \frac{2^{3k/2+\nu+1}\pi^{k/2}}{s^{k+\nu+2}}\Gamma(\nu+1+k/2). \tag{5.6}$$

Using (5.6) in Theorem 4.1, we deduce that

$$\begin{aligned} &\frac{2}{s}\sum_{n=1}^\infty r_k(n)\left(\frac{n}{2}\right)^{(\nu+1)/2} K_{\nu+1}\left(s\sqrt{\frac{n}{2}}\right) + \frac{2^{\nu+1}}{s^{\nu+2}}\Gamma(\nu+1) \\ =& \frac{2^{3k/2+\nu+1}\pi^{k/2}}{s^{k+\nu+2}}\Gamma(\nu+1+k/2) + 2^{3k/2+\nu+1}s^\nu\pi^{k/2}\Gamma(\nu+1+k/2)\sum_{n=1}^\infty \frac{r_k(n)}{(s^2+8\pi^2 n)^{k/2+\nu+1}}. \end{aligned} \tag{5.7}$$

Now, let $s = 2^{3/2}\pi\sqrt{\beta}$. We therefore write (5.7) as

$$\begin{aligned}&\frac{1}{\pi\sqrt{2\beta}}\sum_{n=1}^{\infty} r_k(n)\left(\frac{n}{2}\right)^{(\nu+1)/2} K_{\nu+1}\left(2\pi\sqrt{n\beta}\right) + \frac{2^{\nu+1}}{(2^{3/2}\pi\sqrt{\beta})^{\nu+2}}\Gamma(\nu+1)\\ =&\frac{2^{3k/2+\nu+1}\pi^{k/2}\Gamma(\nu+1+k/2)}{(2^{3/2}\pi\sqrt{\beta})^{k+\nu+2}}\\ &+\frac{2^{3k/2+\nu+1}(2^{3/2}\pi\sqrt{\beta})^{\nu}\pi^{k/2}\Gamma(\nu+1+k/2)}{(8\pi^2)^{k/2+\nu+1}}\sum_{n=1}^{\infty}\frac{r_k(n)}{(\beta+n)^{k/2+\nu+1}}. \end{aligned}\tag{5.8}$$

Multiplying both sides of (5.8) by $2^{1+\nu/2}\pi\sqrt{\beta}$ and simplifying, we arrive at

$$\begin{aligned}&\sum_{n=1}^{\infty} r_k(n)n^{(\nu+1)/2}K_{\nu+1}\left(2\pi\sqrt{n\beta}\right) + \frac{\Gamma(\nu+1)}{2\pi^{\nu+1}\beta^{(\nu+1)/2}}\\ =&\frac{\beta^{(\nu+1)/2}\Gamma(\nu+1+k/2)}{2\pi^{k/2+\nu+1}\beta^{k/2+\nu+1}} + \frac{\beta^{(\nu+1)/2}\Gamma(\nu+1+k/2)}{2\pi^{k/2+\nu+1}}\sum_{n=1}^{\infty}\frac{r_k(n)}{(\beta+n)^{k/2+\nu+1}}. \end{aligned}\tag{5.9}$$

If we define $r_k(0) = 1$ and formally use Lemma 2.3, we find that

$$\lim_{n\to 0} r_k(0)n^{(\nu+1)/2}K_{\nu}(2\pi\sqrt{n\beta}) = \frac{\Gamma(\nu+1)}{2\pi^{\nu+1}\beta^{(\nu+1)/2}}.$$

Hence, we see that we can put (5.9) in the form

$$\sum_{n=0}^{\infty} r_k(n)n^{(\nu+1)/2}K_{\nu+1}\left(2\pi\sqrt{n\beta}\right) = \frac{\beta^{(\nu+1)/2}\Gamma(\nu+1+k/2)}{2\pi^{k/2+\nu+1}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(\beta+n)^{k/2+\nu+1}}. \tag{5.10}$$

From the fact that $r_k(n) = O_k\left(n^{k/2-1+\epsilon}\right)$ for every $\epsilon > 0$, it is clear that the identity (5.10) is valid for $\text{Re}(\nu) > -1$. The identity (5.10) was originally proved by A. I. Popov [14, Eq. (6)]. It was also established in [5, p. 329, Corollary 4.6] by a completely different method. The special case $k = 2$ of this identity was obtained Hardy [13], who used it to prove his famous Ω-theorem for the Gauss circle problem,

6. EXAMPLE 2: $\sigma_k(n)$

Let $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n, where it is assumed that k is an odd positive integer. The generating function for $\sigma_k(n)$ is given by

$$\zeta_k(s) := \zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s}, \qquad \sigma > k+1, \tag{6.1}$$

and it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)\zeta_k(s) = (-1)^{(k+1)/2}(2\pi)^{-(k+1-s)}\Gamma(k+1-s)\zeta_k(k+1-s). \tag{6.2}$$

In the notation of the Dirichlet series and functional equation in (1.2) and (1.3), respectively,

$$a(n) = \sigma_k(n), \quad b(n) = (-1)^{(k+1)/2}\sigma_k(n), \quad \lambda_n = \mu_n = n, \quad \delta = k+1.$$

Assuming that $\text{Re}(\nu), \text{Re}(s) > 0$, apply Theorem 4.1. First, the left-hand side of (4.2) is equal to

$$\frac{2}{s}\sum_{n=1}^{\infty}\sigma_k(n)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n}). \tag{6.3}$$

The first expression on the right-hand side of (4.2) is readily seen to equal

$$2^{3k+\nu+4}s^{\nu}\pi^{k+1}\Gamma(k+\nu+2)\sum_{n=1}^{\infty}\frac{(-1)^{(k+1)/2}\sigma_k(n)}{(s^2+16\pi^2 n)^{k+\nu+2}}. \tag{6.4}$$

It remains to evaluate the integral on the right-hand side of (4.2).

Now $Q_0(s)$ is the sum of the residues of [7, p. 17]

$$R(z) := \frac{\Gamma(z)\zeta(z)\zeta(z-k)x^z}{\Gamma(z+1)}. \tag{6.5}$$

(In Chandrasekaran and Narasimhan's paper [7], they utilize a different convention for Bernoulli numbers, and so our representation for Q_0 takes a different form from theirs.) Observe that $R(z)$ has simple poles at $z = 0, -1, k+1$. Using Euler's formula,

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!},$$

where n is a positive integer and B_n denotes the nth Bernoulli number, we readily find that

$$Q_0(x) = \frac{B_{k+1}}{2(k+1)} - \frac{\delta_{1,k}x}{2} + \frac{(2\pi)^{k+1}(-1)^{(k-1)/2}B_{k+1}x^{k+1}}{2(k+1)\Gamma(k+2)}, \tag{6.6}$$

where

$$\delta_{1,k} = \begin{cases} 1, & \text{if } k=1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} &\int_0^{\infty} Q_0(x)x^{\nu/2}K_{\nu}(s\sqrt{x})dx \\ &= \int_0^{\infty}\left\{\frac{B_{k+1}}{2(k+1)} - \frac{\delta_{1,k}x}{2} + \frac{(2\pi)^{k+1}(-1)^{(k-1)/2}B_{k+1}x^{k+1}}{2(k+1)\Gamma(k+2)}\right\}x^{\nu/2}K_{\nu}(s\sqrt{x})dx \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{6.7}$$

First, as in (5.4), we find that

$$I_1 = \frac{2^{\nu}B_{k+1}\Gamma(\nu+1)}{(k+1)s^{\nu+2}}. \tag{6.8}$$

Secondly, with the use of Lemma 2.5,

$$\begin{aligned} I_2 &= -\frac{\delta_{1,k}}{2}\int_0^{\infty}x^{1+\nu/2}K_{\nu}(s\sqrt{x})dx \\ &= -\frac{\delta_{1,k}}{s^{\nu+4}}\int_0^{\infty}t^{\nu+3}K_{\nu}(t)dt \\ &= -\frac{\delta_{1,k}}{s^{\nu+4}}2^{\nu+2}\Gamma(\nu+2). \end{aligned} \tag{6.9}$$

Thirdly, employing Lemma 2.5, we deduce that

$$\begin{aligned} I_3 =&\frac{(2\pi)^{k+1}(-1)^{(k-1)/2}B_{k+1}}{2(k+1)\Gamma(k+2)}\int_0^\infty x^{k+1+\nu/2}K_\nu(s\sqrt{x})dx\\ =&\frac{(2\pi)^{k+1}(-1)^{(k-1)/2}B_{k+1}}{(k+1)\Gamma(k+2)s^{2k+\nu+4}}\int_0^\infty t^{2k+\nu+3}K_\nu(t)dt\\ =&\frac{2^{3k+\nu+3}\pi^{k+1}(-1)^{(k-1)/2}B_{k+1}}{(k+1)s^{2k+\nu+4}}\Gamma(k+\nu+2). \end{aligned} \tag{6.10}$$

Finally, put (6.8)–(6.10) in (6.7); then substitute (6.3), (6.4), and (6.7) into Theorem 4.1; lastly multiply both sides by $s/2$ to conclude that

$$\begin{aligned} \sum_{n=1}^\infty \sigma_k(n)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n}) =&2^{3k+\nu+3}s^{\nu+1}\pi^{k+1}\Gamma(k+\nu+2)\sum_{n=1}^\infty\frac{(-1)^{(k+1)/2}\sigma_k(n)}{(s^2+16\pi^2n)^{k+\nu+2}}\\ &+\frac{2^{\nu-1}B_{k+1}}{(k+1)s^{\nu+1}}\Gamma(\nu+1)-\frac{\delta_{1,k}}{s^{\nu+3}}2^{\nu+1}\Gamma(\nu+2)\\ &+\frac{2^{3k+\nu+2}\pi^{k+1}(-1)^{(k-1)/2}B_{k+1}}{(k+1)s^{2k+\nu+3}}\Gamma(k+\nu+2). \end{aligned} \tag{6.11}$$

We now put (6.11) in a more palatable form. From (6.2),

$$\zeta_k(0)=\zeta(0)\zeta(-k)=-\frac{1}{2}\cdot-\frac{B_{k+1}}{k+1}=\frac{B_{k+1}}{2(k+1)},$$

by [11, p. 12]. Define

$$\sigma_k(0)=-\zeta_k(0)=-\frac{B_{k+1}}{2(k+1)}. \tag{6.12}$$

Next, define a term for $n=0$ on the left-hand side of (6.11) by formally using Lemma 2.3. Therefore, with also the use of (6.12),

$$\lim_{n\to0}\sigma_k(0)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n})=-\frac{2^{\nu-1}B_{k+1}\Gamma(\nu+1)}{(k+1)s^{\nu+1}}. \tag{6.13}$$

Hence, utilizing (6.13) in (6.11), we have shown that

$$\begin{aligned} &\sum_{n=0}^\infty \sigma_k(n)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n})\\ =&2^{3k+\nu+3}s^{\nu+1}\pi^{k+1}\Gamma(k+\nu+2)\sum_{n=1}^\infty\frac{(-1)^{(k+1)/2}\sigma_k(n)}{(s^2+16\pi^2n)^{k+\nu+2}}\\ &-\frac{\delta_{1,k}}{s^{\nu+3}}2^{\nu+1}\Gamma(\nu+2)+\frac{2^{3k+\nu+2}\pi^{k+1}(-1)^{(k-1)/2}B_{k+1}}{(k+1)s^{2k+\nu+3}}\Gamma(k+\nu+2). \end{aligned} \tag{6.14}$$

Note that the term for $n=0$ on the right-hand side of (6.14) is equal to

$$\begin{aligned} &\frac{2^{3k+\nu+3}s^{\nu+1}\pi^{k+1}\Gamma(k+\nu+2)(-1)^{(k+1)/2}\sigma_k(0)}{s^{2k+2\nu+4}}\\ =&\frac{2^{3k+\nu+2}\pi^{k+1}\Gamma(k+\nu+2)(-1)^{(k-1)/2}B_{k+1}}{(k+1)s^{2k+\nu+3}}, \end{aligned} \tag{6.15}$$

by (6.12). Using (6.15) in (6.14), we conclude that

$$\sum_{n=0}^{\infty}\sigma_k(n)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n})+\frac{\delta_{1,k}}{s^{\nu+3}}2^{\nu+1}\Gamma(\nu+2)$$
$$=2^{3k+\nu+3}s^{\nu+1}\pi^{k+1}\Gamma(k+\nu+2)\sum_{n=0}^{\infty}\frac{(-1)^{(k+1)/2}\sigma_k(n)}{(s^2+16\pi^2n)^{k+\nu+2}}.$$

Note that the elementary bound $\sigma_k(n)=O(n^{k+\epsilon})$ implies that the identity above is actually valid for $\mathrm{Re}(\nu)>-1$. Letting $\nu=-1/2$ and using (1.1) leads to the following result of Chandrasekharan and Narasimhan [7, Equation (60)]:

$$\sum_{n=1}^{\infty}\sigma_k(n)e^{-s\sqrt{n}}=2^{3k+3}\Gamma\left(k+\frac{3}{2}\right)\pi^{k+\frac{1}{2}}\sum_{n=1}^{\infty}\frac{s(-1)^{(k+1)/2}\sigma_k(n)}{(s^2+16\pi^2n)^{k+3/2}}$$
$$+\frac{B_{k+1}}{2(k+1)}-\frac{\delta_{1,k}}{s^2}+\frac{2^{3k+2}\pi^{k+\frac{1}{2}}(-1)^{(k-1)/2}B_{k+1}}{(k+1)s^{2k+2}}\Gamma\left(k+\frac{3}{2}\right). \quad (6.16)$$

7. Example 3: $\tau(n)$

Recall that the Dirichlet series for Ramanujan's arithmetical function $\tau(n)$

$$f(s):=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s},\quad \sigma>\frac{13}{2},$$

satisfies the functional equation

$$\chi(s):=(2\pi)^{-s}\Gamma(s)f(s)=(2\pi)^{-(12-s)}\Gamma(12-s)f(12-s). \quad (7.1)$$

The function $\chi(s)$ is an entire function, and so $Q_0(x)\equiv 0$. Clearly, $\lambda_n=\mu_n=n$ and $\delta=12$. Thus, for $\mathrm{Re}(\nu),\mathrm{Re}(s)>0$, from Theorem 4.1 we can immediately deduce the identity

$$\sum_{n=1}^{\infty}\tau(n)n^{(\nu+1)/2}K_{\nu+1}(s\sqrt{n})=2^{36+\nu}s^{\nu+1}\pi^{12}\Gamma(13+\nu)\sum_{n=1}^{\infty}\frac{\tau(n)}{(s^2+16\pi^2n)^{\nu+13}}. \quad (7.2)$$

Let $\nu=-\frac{1}{2}$. Then, from (1.1),

$$K_{1/2}(s\sqrt{n})=\left(\frac{\pi}{2s\sqrt{n}}\right)^{1/2}e^{-s\sqrt{n}}. \quad (7.3)$$

Hence, by (7.2) and (7.3),

$$\sum_{n=1}^{\infty}\tau(n)e^{-s\sqrt{n}}=2^{36}\pi^{23/2}\Gamma\left(\frac{25}{2}\right)\sum_{n=1}^{\infty}\frac{s\tau(n)}{(s^2+16\pi^2n)^{25/2}}. \quad (7.4)$$

The identity (7.4) is originally due to Chandrasekharan and Narasimhan [7, p. 16, Eq. (56)].

8. A THEOREM OF G. N. WATSON

In this section, we obtain a theorem of Watson [15, Equation (4)] as a special case of Theorem 4.1, namely, for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\nu) > 0$,

$$\begin{aligned}\frac{1}{2}\Gamma(\nu) + 2\sum_{n=1}^{\infty}\left(\frac{1}{2}nz\right)^{\nu} K_{\nu}(nz) \\ = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right) z^{2\nu}\left\{\frac{1}{z^{2\nu+1}} + 2\sum_{n=1}^{\infty}\frac{1}{(z^2+4\pi^2n^2)^{\nu+1/2}}\right\}. \qquad (8.1)\end{aligned}$$

The functional equation of the Riemann zeta function is given by [11, p. 14]

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s). \qquad (8.2)$$

Hence, replacing s by $2s$, we see that it can be converted into the form in (1.3) with $\delta = 1/2$. Therefore, we invoke Theorem 4.1 with $a(n) = b(n) = 1$, $\lambda_n = \mu_n = n^2/2$, whence

$$\begin{aligned}\frac{2}{s}\sum_{n=1}^{\infty}(n/\sqrt{2})^{\nu+1}K_{\nu+1}(sn/\sqrt{2}) = 2^{\nu+5/2}s^{\nu}\pi^{1/2}\Gamma(\nu+3/2)\sum_{n=1}^{\infty}\frac{1}{(s^2+8\pi^2n^2)^{\nu+3/2}} \\ + \int_0^{\infty} Q_0(x)x^{\nu/2}K_{\nu}(s\sqrt{x})\,dx. \qquad (8.3)\end{aligned}$$

Here $Q_0(x) = -1/2 + \sqrt{2x}$, the sum of the residues of $\zeta(2z)(2x)^z/z$ at 0 and $1/2$, so that

$$\int_0^{\infty} Q_0(x)x^{\nu/2}K_{\nu}(s\sqrt{x})\,dx = 2^{\nu+3/2}\sqrt{\pi}s^{-\nu-3}\Gamma(\nu+3/2) - 2^{\nu}s^{-\nu-2}\Gamma(1+\nu). \qquad (8.4)$$

From (8.3) and (8.4),

$$\begin{aligned}\frac{2^{(1-\nu)/2}}{s}\sum_{n=1}^{\infty}(n/\sqrt{2})^{\nu+1}K_{\nu+1}(sn/\sqrt{2}) = 2^{\nu+5/2}s^{\nu}\pi^{1/2}\Gamma(\nu+3/2)\sum_{n=1}^{\infty}\frac{1}{(s^2+8\pi^2n^2)^{\nu+3/2}} \\ + 2^{\nu+3/2}\sqrt{\pi}s^{-\nu-3}\Gamma(\nu+3/2) - 2^{\nu}s^{-\nu-2}\Gamma(1+\nu). \\ (8.5)\end{aligned}$$

Next let $s = z\sqrt{2}$ in the foregoing equation, then multiply the resulting identity by $2^{-\nu/2}z^{\nu+2}$, and replace ν by $\nu - 1$ to arrive at (8.1) upon simplification.

9. PRIMITIVE CHARACTERS $\chi(n)$

Let χ denote a primitive character modulo q. Because the functional equations for the Dirichlet L-series

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \qquad \sigma > 0,$$

are different for χ even and χ odd, we separate the two cases.

Suppose first that χ is odd. Then the functional equation for $L(s,\chi)$ is given by [10, p. 71]

$$\chi(s) := \left(\frac{\pi}{q}\right)^{-s}\Gamma(s)L(2s-1,\chi) = -\frac{i\tau(\chi)}{\sqrt{q}}\left(\frac{\pi}{q}\right)^{-(\frac{3}{2}-s)}\Gamma\left(\tfrac{3}{2}-s\right)L(2-2s,\overline{\chi}), \quad (9.1)$$

where $\overline{\chi}(n)$ denotes the complex conjugate of $\chi(n)$, and $\tau(\chi)$ denotes the Gauss sum

$$\tau(\chi) := \sum_{n=1}^{q} \chi(n) e^{2\pi i n/q}.$$

Hence, in the notation of (1.2) and (1.3),

$$a(n) = n\chi(n), \quad b(n) = -\frac{i\tau(\chi)}{\sqrt{q}} n\overline{\chi}(n), \quad \lambda_n = \mu_n = \frac{n^2}{2q}, \quad \delta = \frac{3}{2}.$$

Also, $\chi(s)$ is an entire function, and consequently $Q_0(x) \equiv 0$. Hence, by Theorem 4.1, for $\mathrm{Re}(\nu), \mathrm{Re}(s) > 0$,

$$\frac{2}{s}\sum_{n=1}^{\infty} n\chi(n) \left(\frac{n^2}{2q}\right)^{(\nu+1)/2} K_{\nu+1}\left(s\sqrt{\frac{1}{2q}}n\right)$$
$$= -\frac{i\tau(\chi)}{\sqrt{q}} 2^{\nu+11/2} s^{\nu} \pi^{3/2} \Gamma\left(\nu + \tfrac{5}{2}\right) \sum_{n=1}^{\infty} \frac{n\overline{\chi}(n)}{(s^2 + 8\pi^2 n^2/q)^{\nu+5/2}}. \tag{9.2}$$

Now multiply both sides of (9.2) by $\frac{1}{2}s\,(2q)^{(\nu+1)/2}$, and then let $s = \sqrt{2q}\,r$ to deduce that

$$\sum_{n=1}^{\infty} \chi(n) n^{\nu+2} K_{\nu+1}(rn) = -i\tau(\chi) \frac{r^{\nu+1} q^{2\nu+3} \Gamma\left(\nu + \frac{5}{2}\right)}{2^{\nu+2}\pi^{2\nu+7/2}} \sum_{n=1}^{\infty} \frac{n\overline{\chi}(n)}{(n^2 + q^2 r^2/(4\pi^2))^{\nu+5/2}}. \tag{9.3}$$

Second, let χ be even. Then the functional equation of $L(s,\chi)$ is given by [10, p. 69]

$$\chi(s) := \left(\frac{\pi}{q}\right)^{-s} \Gamma(s) L(2s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{\pi}{q}\right)^{-(\frac{1}{2}-s)} \Gamma\left(\tfrac{1}{2} - s\right) L(1-2s, \overline{\chi}), \tag{9.4}$$

Hence, by (1.2) and (1.3),

$$a(n) = \chi(n), \quad b(n) = \frac{\tau(\chi)}{\sqrt{q}} \overline{\chi}(n), \quad \lambda_n = \mu_n = \frac{n^2}{2q}, \quad \delta = \frac{1}{2}.$$

Also, $\chi(s)$ is an entire function, and consequently $Q_0(x) \equiv 0$. Hence, by Theorem 4.1, for $\mathrm{Re}(\nu), \mathrm{Re}(s) > 0$,

$$\frac{2}{s}\sum_{n=1}^{\infty} \chi(n) \left(\frac{n^2}{2q}\right)^{(\nu+1)/2} K_{\nu+1}\left(s\sqrt{\frac{1}{2q}}n\right)$$
$$= \frac{\tau(\chi)}{\sqrt{q}} 2^{\nu+5/2} s^{\nu} \pi^{1/2} \Gamma\left(\nu + \tfrac{3}{2}\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{(s^2 + 8\pi^2 n^2/q)^{\nu+3/2}}. \tag{9.5}$$

Multiply both sides of (9.2) by $\frac{1}{2}s(2q)^{(\nu+1)/2}$ and then let $s = \sqrt{2q}\,r$ to obtain

$$\sum_{n=1}^{\infty} \chi(n) n^{\nu+1} K_{\nu+1}(rn) = \tau(\chi) \frac{r^{\nu+1} q^{2(\nu+1)} \Gamma\left(\nu + \frac{3}{2}\right)}{2^{\nu+2}\pi^{2\nu+5/2}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{(n^2 + q^2 r^2/(4\pi^2))^{\nu+3/2}}. \tag{9.6}$$

Identities (9.3) and (9.6) were first obtained in [4, Theorem 2.1] and are character analogues of (8.1).

10. Ideal Functions $F(n)$ of Imaginary Quadratic Fields

Let $F(n)$ denote the number of integral ideals of norm n in an imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$, where D is the discriminant of K. Then the Dedekind zeta function

$$\zeta_K(s) := \sum_{n=1}^{\infty} \frac{F(n)}{n^s}, \qquad \sigma > 1,$$

satisfies the functional equation

$$\left(\frac{2\pi}{\sqrt{D}}\right)^{-s} \Gamma(s)\zeta_K(s) = \left(\frac{2\pi}{\sqrt{D}}\right)^{s-1} \Gamma(1-s)\zeta_K(1-s). \tag{10.1}$$

We note from (1.2) and (1.3) that

$$a(n) = b(n) = F(n), \qquad \lambda_n = \mu_n = n/\sqrt{D}, \qquad \delta = 1.$$

The function $\zeta_K(s)$ has an analytic continuation to the entire complex plane where it is analytic except for a simple pole at $s = 1$. From [9, p. 212],

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2\pi h(K)R(K)}{w(K)\sqrt{D}}, \tag{10.2}$$

where $h(K), R(K)$, and $w(K)$ denote, respectively, the class number of K, the regulator of K, and the number of roots of unity in K. Furthermore, from (10.1) and (10.2),

$$\zeta_K(0) = \lim_{s\to 0} \frac{\sqrt{D}}{2\pi} \cdot \frac{1}{s\Gamma(s)} \cdot s\zeta_K(1-s) = \frac{\sqrt{D}}{2\pi} \cdot -\frac{2\pi h(K)R(K)}{w(K)\sqrt{D}} = -\frac{h(K)R(K)}{w(K)}. \tag{10.3}$$

For simplicity, set $d = \sqrt{D}, h = h(K), R = R(K)$, and $w = w(K)$. From (10.3) and (10.2),

$$\begin{aligned} Q_0(x) =& \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(z)}{\Gamma(z+1)} d^z \zeta_K(z) x^z dz \\ =& -\frac{hR}{w} + \frac{2\pi hRx}{w}. \end{aligned} \tag{10.4}$$

We apply Theorem 4.1. First, we calculate the integral on the right-hand side of (4.2). To that end, by (10.4) and (10.3),

$$\begin{aligned} I :=& \int_0^{\infty} Q_0(x) x^{\nu/2} K_\nu(s\sqrt{x}) dx \\ =& -\frac{hR}{w} \int_0^{\infty} x^{\nu/2} K_\nu(s\sqrt{x}) dx + \frac{2\pi hR}{w} \int_0^{\infty} x^{\nu/2+1} K_\nu(s\sqrt{x}) dx \\ =& -\frac{hR}{w} \frac{2^{\nu+1}}{s^{\nu+2}} \Gamma(\nu+1) + \frac{2\pi hR}{w} \frac{2^{\nu+3}}{s^{\nu+4}} \Gamma(\nu+2), \end{aligned} \tag{10.5}$$

by Lemmas 2.4 and 2.5. Hence, with the use of (10.5), Theorem 4.1 yields

$$\begin{aligned} \sum_{n=1}^{\infty} F(n)(n/d)^{(\nu+1)/2} K_{\nu+1}(s\sqrt{n/d}) = 2^{\nu+3} s^{\nu+1} \pi \Gamma(\nu+2) \sum_{n=1}^{\infty} \frac{F(n)}{(s^2+16\pi^2 n/d)^{\nu+2}} \\ - \frac{hR}{w} \frac{2^\nu}{s^{\nu+1}} \Gamma(\nu+1) + \frac{2\pi hR}{wd} \frac{2^{\nu+2}}{s^{\nu+3}} \Gamma(\nu+2). \end{aligned} \tag{10.6}$$

Let $s = 4\pi\sqrt{r/d}$ and multiply both sides by $d^{(\nu+1)/2}$. Hence,

$$\sum_{n=1}^{\infty} F(n)n^{(\nu+1)/2}K_{\nu+1}(4\pi\sqrt{rn}/d) = \frac{1}{2\sqrt{r}}\left(\frac{d\sqrt{r}}{2\pi}\right)^{\nu+2}\Gamma(\nu+2)\sum_{n=1}^{\infty}\frac{F(n)}{(r+n)^{\nu+2}}$$
$$-\frac{hR}{2w}\left(\frac{d}{2\pi\sqrt{r}}\right)^{\nu+1}\Gamma(\nu+1) + \frac{hR}{2w\sqrt{r}}\left(\frac{d}{2\pi\sqrt{r}}\right)^{\nu+2}\Gamma(\nu+2). \tag{10.7}$$

From a formal use of Lemma 2.3,

$$\lim_{n\to 0} n^{(\nu+1)/2}K_{\nu+1}\left(\frac{4\pi\sqrt{rn}}{d}\right) = \frac{1}{2}\left(\frac{d}{2\pi\sqrt{r}}\right)^{\nu+1}\Gamma(\nu+1). \tag{10.8}$$

Hence, if we define $F(0) = hR/w$ and then note that $F(0)$ multiplied by the right side of (10.8) appears on the right side of (10.7), we can rewrite (10.7) in the form

$$\sum_{n=0}^{\infty} F(n)n^{(\nu+1)/2}K_{\nu+1}(4\pi\sqrt{rn}/d) = \frac{1}{2\sqrt{r}}\left(\frac{d\sqrt{r}}{2\pi}\right)^{\nu+2}\Gamma(\nu+2)\sum_{n=0}^{\infty}\frac{F(n)}{(r+n)^{\nu+2}}.$$

From [8, Lemma 9], we see that $F(n) = O(n^{\epsilon})$ for every $\epsilon > 0$. Hence the foregoing identity is actually valid for Re$(\nu) > -1$. Hence, letting $\nu = -1/2$, we obtain the special case

$$\sum_{n=0}^{\infty} F(n)e^{-4\pi\sqrt{rn}/d} = \frac{d^{3/2}\sqrt{r}}{4\pi}\sum_{n=0}^{\infty}\frac{F(n)}{(r+n)^{3/2}}.$$

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