APPLICATIONS OF THE LIPSCHITZ SUMMATION FORMULA AND A GENERALIZATION OF RAABE'S COSINE TRANSFORM

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Abstract. General summation formulas have been proved to be very useful in analysis, number theory and other branches of mathematics. The Lipschitz summation formula is one of them. In this paper, we give its application by providing a new transformation formula which generalizes that of Ramanujan. Ramanujan's result, in turn, is a generalization of the modular transformation of Eisenstein series $E_k(z)$ on $\mathrm{SL}_2(\mathbb{Z})$, where $z \to -1/z, z \in \mathbb{H}$. The proof of our result involves delicate analysis containing Cauchy Principal Value integrals. A simpler proof of a recent result of ours with Kesarwani giving a non-modular transformation for $\sum_{n=1}^{\infty} \sigma_{2m}(n) e^{-ny}$ is also derived using the Lipschitz summation formula. In the pursuit of obtaining this transformation, we naturally encounter a new generalization of Raabe's cosine transform whose several properties are also demonstrated. As a corollary of this result, we get a generalization of Wright's asymptotic estimate for the generating function of the number of plane partitions of a positive integer n.

Contents

1. Introduction

Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}\backslash\{0\}$. Ramanujan's famous formula for $\zeta(2m+1)$ is given by[1] [42, p. 173, Ch. 14, Entry 21(i)], [41, p. 319-320, formula (28)], [6, p. 275-276]

$$\alpha^{-m}\left\{\frac{1}{2}\zeta(2m+1)+\sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\alpha}-1}\right\}=(-\beta)^{m}\left\{\frac{1}{2}\zeta(2m+1)+\sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2n\beta}-1}\right\}$$

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[1] Ramanujan's formula is actually valid for any complex α, β such that $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ and $\alpha\beta = \pi^2$.

$$- 2^{2m}\sum_{k=0}^{m}\frac{(-1)^k B_{2k}B_{2m+2-2k}}{(2k)!(2m+2-2k)!}\alpha^{m+1-k}\beta^k, \tag{1.1}$$

where, as customary, $\zeta(s)$ denotes the Riemann zeta function and B_n denotes the n^{th} Bernoulli number. The above formula has received enormous attention from several mathematicians over the years and has been rediscovered many times, for example, see [23, Theorem 9] and [33]. It is an impressive result, for, it encapsulates not only the transformation formulas of the Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ and the corresponding ones for their Eichler integrals but also the transformation property of the logarithm of the Dedekind eta function. For a delightful historical account on it, we refer the reader to the excellent survey [8]. There are several generalizations of (1.1) in the literature, for example, [11], [12], [14], [16], [17], [29] and [24]. In his second notebook [42][p. 269], Ramanujan himself provided the following generalization of (1.1).

Let α and β be two positive real numbers such that $\alpha\beta = 4\pi^2$. Then for $\mathrm{Re}(s) > 2$, we have

$$\alpha^{s/2}\left\{\frac{\Gamma(s)\zeta(s)}{(2\pi)^s} + \cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{n^{s-1}}{e^{n\alpha}-1}\right\} = \beta^{s/2}\left\{\cos\left(\frac{\pi s}{2}\right)\frac{\Gamma(s)\zeta(s)}{(2\pi)^s} + \sum_{n=1}^{\infty}\frac{n^{s-1}}{e^{n\beta}-1}\right.$$
$$\left. - \sin\left(\frac{\pi s}{2}\right)\mathrm{PV}\int_0^{\infty}\frac{x^{s-1}}{e^{2\pi x}-1}\cot\left(\frac{1}{2}\beta x\right)dx\right\}, \tag{1.2}$$

where PV denotes the principal value integral. The above formula has been proved in [7, p. 416]. Also see [9, Section 9] for a recent generalization of (1.2).

Unfortunately, Ramanujan's formula (1.2) has not received as much attention as (1.1). But it is also a noteworthy result because it not only gives the transformation formula for the Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ in the special case $s = 2m, m \in \mathbb{N}, m > 1$, but also reveals the obstruction to modularity for other values of s, which is evident due to the appearance of the integral on its right-hand side. Note that the last term involving the integral disappears for $s = 2m$.

One of the goals of this paper is to derive a generalization of (1.2):

Theorem 1.1. *Let $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ such that $\alpha\beta = 4\pi^2$. Let $0 \leq a < 1$. Then, for $\mathrm{Re}(s) > 2$, the following transformation holds*

$$\alpha^{s/2}\left\{\frac{\Gamma(s)\zeta(s)}{(2\pi)^s} + \frac{1}{2}\sum_{n=1}^{\infty}n^{s-1}\left(\frac{e^{\pi i s/2}}{e^{n\alpha-2\pi i a}-1} + \frac{e^{-\pi i s/2}}{e^{n\alpha+2\pi i a}-1}\right)\right\}$$
$$= \beta^{s/2}\left\{\frac{\Gamma(s)}{(2\pi)^s}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi s}{2}+2\pi a k\right)}{k^s} + \sum_{n=1}^{\infty}\frac{(n-a)^{s-1}}{e^{(n-a)\beta}-1}\right.$$
$$\left. -\frac{1}{2i}\mathrm{PV}\int_0^{\infty}x^{s-1}\left(\frac{e^{\pi i s/2}}{e^{2\pi x-2\pi i a}-1} - \frac{e^{-\pi i s/2}}{e^{2\pi x+2\pi i a}-1}\right)\cot\left(\frac{1}{2}\beta x\right)dx\right\}. \tag{1.3}$$

The above theorem reduces to Ramanujan's formula (1.2) for $a = 0$. Also for $s = 2m, m \in \mathbb{N}$ and $a = 0$, it gives (1.1).

A special case of Theorem 1.1 is the new transformation given below.

Corollary 1.2. *Let $m \in \mathbb{N}$. For $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ such that $\alpha\beta = 4\pi^2$, we have*

$$\alpha^m \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{n\alpha}+1} + (-\beta)^m \sum_{n=1}^{\infty} \frac{(n-1/2)^{2m-1}}{e^{(n-1/2)\beta}+1} = -\left\{\alpha^m - (2^{1-2m}-1)(-\beta)^m\right\} \frac{B_{2m}}{4m}. \tag{1.4}$$

Equation (1.4) is a "hybrid" analogue of the following transformation formula for the Eisenstein series over $\mathrm{SL}_2(\mathbb{Z})$ in that the role of n in the first series is played by $n - 1/2$ in the second.

$$\alpha^m \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{n\alpha}-1} - (-\beta)^m \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{n\beta}-1} = \{\alpha^m - (-\beta)^m\} \frac{B_{2m}}{4m}.$$

As an application of Corollary 1.2, we obtain closed-form evaluations of two infinite series, which, to the best of our knowledge, are new.

Corollary 1.3. *For any odd positive integer m greater than 1,*

$$\sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{2n\pi}+1} = (1-2^{2-2m})\frac{B_{2m}}{4m},$$

$$\text{and} \quad \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{4n\pi}-1} = 2^{-1-2m}\frac{B_{2m}}{m}.$$

First, let $s = 2m$ and $a = 1/4$ in (1.3), then let $s = 2m$ and $a = 3/4$ in (1.3), and add the corresponding sides of the resulting identities. This leads to the transformation between just the infinite series which we record below in (1.5). Similarly subtracting the corresponding sides of the two resulting identities expresses a principal value integral in terms of a Lambert series, which is given in (1.6).

Corollary 1.4. *Let $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ such that $\alpha\beta = 4\pi^2$. For $m \in \mathbb{N}, m > 1$,*

$$\begin{aligned}
&\alpha^m 2^{4m-1} \left\{\frac{\Gamma(2m)\zeta(2m)}{(2\pi)^{2m}} + (-1)^{m+1} \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{2n\alpha}+1}\right\} \\
&= \beta^m \left\{(-1)^{m+1} \frac{\Gamma(2m)\zeta(2m)(2^{2m-1}-1)}{(2\pi)^{2m}} + \sum_{n=1}^{\infty} \frac{(4n-1)^{2m-1}}{e^{(4n-1)\beta/4}-1} + \sum_{n=1}^{\infty} \frac{(4n-3)^{2m-1}}{e^{(4n-3)\beta/4}-1}\right\},
\end{aligned} \tag{1.5}$$

and,

$$\mathrm{PV}\int_0^{\infty} \mathrm{sech}(x) \cot(2\beta x)\, x^{2m-1} dx = (-1)^{m+1} 4^{1-2m} \sum_{n=1}^{\infty} \frac{\chi(n) n^{2m-1}}{e^{n\beta}-1}, \tag{1.6}$$

where $\chi(n)$ is a Dirichlet character modulo 4 given by

$$\chi(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4, \\ 0, & \text{if } n \equiv 0, 2 \pmod 4. \end{cases} \tag{1.7}$$

We note that the series on the right-hand side of (1.6) cannot be treated using [11, Theorem 1].

We now transition towards to the second goal of our paper. Recently, the current authors, along with Kesarwani [16], extensively studied a more general Lambert series

$$\sum_{n=1}^{\infty} \frac{n^s}{e^{ny}-1} = \sum_{n=1}^{\infty} \sigma_s(n) e^{-ny} \quad (s \in \mathbb{C},\ \mathrm{Re}(y) > 0) \tag{1.8}$$

than the ones appearing in (1.1). Here $\sigma_s(n) := \sum_{d|n} d^s$ is the generalized divisor function. Among other things, they obtained [16, Theorem 2.5] an explicit transformation for any $\mathrm{Re}(s) > -1$ and $\mathrm{Re}(y) > 0$, which is given next.

$$\sum_{n=1}^{\infty}\sigma_s(n)e^{-ny}+\frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+s}\operatorname{cosec}\left(\frac{\pi s}{2}\right)+1\right)\zeta(-s)-\frac{1}{y}\zeta(1-s)$$
$$=\frac{2\pi}{y\sin\left(\frac{\pi s}{2}\right)}\sum_{n=1}^{\infty}\sigma_s(n)\left(\frac{(2\pi n)^{-s}}{\Gamma(1-s)}{}_1F_2\left(1;\frac{1-s}{2},1-\frac{s}{2};\frac{4\pi^4n^2}{y^2}\right)-\left(\frac{2\pi}{y}\right)^{s}\cosh\left(\frac{4\pi^2n}{y}\right)\right), \tag{1.9}$$

where ${}_1F_2(a;b,c;z) := \sum_{n=0}^{\infty}\frac{(a)_nz^n}{(b)_n(c)_nn!},\ z\in\mathbb{C},\ (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ is the generalized hypergeometric function.

The explicit transformations of the type (1.9) are always desirable due to their possible applications in analytic number theory, especially in the theory of zeta functions. See the recent paper [3] for a beautiful application of (1.9) in the theory of $\zeta(s)$ by applying the operator $\frac{d}{ds}\big|_{s=0}$ on both sides, thereby resulting in a transformation of the Lambert series of the logarithm, that is, $\displaystyle\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1}$.

The authors of [16, Theorem 2.5] also analytically continued (1.9) to $\mathrm{Re}(s) > -2m-3,\ m\in\mathbb{N}\cup\{0\}$. Then, as a special case, they not only obtained Ramanujan's formula (1.1) and the transformation formula of the logarithm of the Dedekind eta function but also new transformations when s is an even integer. For example, they established an explicit result [16, Theorem 2.11] for (1.8) when $s=2m,\ m>0$. We record it in the following theorem. It comprises two special functions $\mathrm{Shi}(z)$ and $\mathrm{Chi}(z)$, known as the hyperbolic sine and cosine integrals, respectively defined by [36, p. 150, Equation (6.2.15), (6.2.16)]

$$\mathrm{Shi}(z) := \int_0^z\frac{\sinh(t)}{t}\,dt,\quad \mathrm{Chi}(z) := \gamma+\log(z)+\int_0^z\frac{\cosh(t)-1}{t}\,dt, \tag{1.10}$$

where γ is Euler's constant.

Theorem 1.5. *Let $m\in\mathbb{N}$. Then for $\mathrm{Re}(y)>0$, we have*

$$\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny}-\frac{(2m)!}{y^{2m+1}}\zeta(2m+1)+\frac{B_{2m}}{2my}=(-1)^m\frac{2}{\pi}\left(\frac{2\pi}{y}\right)^{2m+1}\sum_{n=1}^{\infty}\sigma_{2m}(n)\Bigg\{\sinh\left(\frac{4\pi^2n}{y}\right)\mathrm{Shi}\left(\frac{4\pi^2n}{y}\right)$$
$$-\cosh\left(\frac{4\pi^2n}{y}\right)\mathrm{Chi}\left(\frac{4\pi^2n}{y}\right)+\sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2n}{y}\right)^{-2j}\Bigg\}. \tag{1.11}$$

The modular transformation for $\sum_{n=1}^{\infty}\sigma_{2m+1}(n)e^{-ny}$ transforms it into

$$\sum_{n=1}^{\infty}\sigma_{2m+1}(n)e^{-4\pi^2n/y}=-\sum_{n=1}^{\infty}\sigma_{2m+1}(n)\left\{\sinh\left(\frac{4\pi^2n}{y}\right)-\cosh\left(\frac{4\pi^2n}{y}\right)\right\}. \tag{1.12}$$

In view of this, it is important to note that while going from $s=2m+1$ to $s=2m$ in $\sum_{n=1}^{\infty}\sigma_s(n)e^{-ny}$, the expression $\sinh\left(\frac{4\pi^2n}{y}\right)-\cosh\left(\frac{4\pi^2n}{y}\right)$ in (1.12) is to be replaced by the corresponding one on the right-hand side of (1.11). (Note that the finite sum $\sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2n}{y}\right)^{-2j}$ in the summand of the series on the right-hand side of (1.12) is essential for its convergence; for details, see the proof of Theorem 1.5 in Section 4.)

Theorem 1.5 readily gives the following asymptotic estimate for $\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny}$:

Corollary 1.6. *Let $m \in \mathbb{N}$. As $y \to 0$ in $|\arg(y)| < \pi/2$,*

$$\sum_{n=1}^{\infty} \sigma_{2m}(n) e^{-ny} = \frac{(2m)!}{y^{2m+1}} \zeta(2m-1) - \frac{B_{2m}}{2my} - \frac{2}{\pi(2\pi)^{2m-1}} \sum_{j=1}^{r+1} \frac{\Gamma(2m+2j)\zeta(2m+2j)\zeta(2j)}{(2\pi)^{4j}} y^{2j-1} + O\left(y^{2r+3}\right). \tag{1.13}$$

The case $m = 1$ of the series on the left-hand side of (1.11) (or (1.6)) has the following interesting connection with the generation function for plane partitions studied by MacMahon [1, p. 184]:

$$\sum_{n=1}^{\infty} \sigma_2(n) e^{-ny} = x \frac{d}{dx} \log(F(x)), \tag{1.14}$$

where $F(x) := \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^n}$, with $y = \log(1/x)$, where $|x| < 1$. In his work on finding the asymptotic estimate of $q(n)$, the number of plane partitions of a positive integer n, Wright [46, Lemma 1] first found the asymptotic estimate of $F(x)$ as $x \to 1^-$. His result on $F(x)$ follows readily from our Corollary 1.6 and is rephrased in the following corollary.

Corollary 1.7. *As $x \to 1^-$, we have*

$$F(x) = e^c (\log x)^{1/12} \exp\left(\frac{\zeta(3)}{\log^2 x}\right) \exp\left(\sum_{j=1}^{r+1} \delta_j (\log x)^{2j}\right) \left(1 + O_r\left((\log x)^{2r+4}\right)\right), \tag{1.15}$$

where, c is a constant, and

$$\delta_j := \frac{\Gamma(2j+2)\zeta(2j+2)\zeta(2j)}{2\pi^2 j (2\pi)^{4j}}. \tag{1.16}$$

It is important to note that Wright obtained the above result through a long calculation (see [46, pp. 180-184] whereas it is a trivial consequence of our Corollary 1.6 as shown in Section 5. On the other hand, the advantage of his method is that it gives a representation of the constant c in terms of an integral, namely, $c = 2 \int_0^{\infty} \frac{y \log(y)}{e^{2\pi y} - 1}\, dy$.

The Lambert series (1.8), whose special cases were considered in (1.1) and (1.5), has been studied by many mathematicians over the years. For a detailed survey, see [16]. One of the earliest mathematicians to study it was Wigert, who wrote several papers on this subject. In [45], Wigert examined the Lambert series when $0 < s < 1$. Later, Kuylenstierna [30] provided a simple proof of Wigert's result using double zeta $\zeta_2(s, \tau) := \sum_{m,n=0}^{\infty} \frac{1}{(m+n\tau)^s}$, $\mathrm{Re}(s) > 2$, $\tau \in \mathbb{C} \backslash (-\infty, 0]$. However, both of them were interested only in the asymptotics of the series in (1.8), not in explicit transformations. In his work, Kuylenstierna essentially uses the Lipschitz summation formula [32]:

Theorem 1.8. *Let $0 \leq a < 1$. Then for $\mathrm{Re}(s) > 1$ and $\tau \in \mathbb{H}$, we have*

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i \tau (n-a)}}{(n-a)^{1-s}} = \frac{\Gamma(s)}{(-2\pi i)^s} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i a k}}{(k+\tau)^s}. \tag{1.17}$$

The Lipschitz summation formula has several nice applications and generalizations, for example, see [5, 28, 37]. Theorem 1.8 is usually proved using Poisson summation formula, for example, see [40, p. 77–79]. For other proofs, one can look at the paper of Vági [44].

It does not seem to be easy to get Theorem 1.5 as a special case of Ramanujan's formula (1.2) or our generalization (1.3), because one has to transform the Lambert series and the principal value integral on the right-hand side of (1.2) into the series in (1.11) involving the special functions $\text{Shi}(z)$ and $\text{Chi}(z)$.

In this paper, we prove Theorem 1.1 and Theorem 1.5 using the Lipschitz summation formula (1.17). The proof of Theorem 1.5 through this approach involves a nice generalization of the following identity [14, Theorem 2.2]

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{t\cos(t)}{t^2+n^2u^2}\,dt=\frac{1}{2}\left\{\log\left(\frac{u}{2\pi}\right)-\frac{1}{2}\left(\psi\left(\frac{iu}{2\pi}\right)+\psi\left(-\frac{iu}{2\pi}\right)\right)\right\}, \tag{1.18}$$

where $\text{Re}(u)>0$, and $\psi(z):=\Gamma'(z)/\Gamma(z)$ is the digamma function. In [14, Theorem 2.4], the above identity was employed to obtain a two-parameter generalization of (1.1). Various applications of (1.18) can be found in [15, 16].

Observe that the summand of the left-hand side of (1.18) is the Raabe cosine transform defined for $\text{Re}(w)>0$ and $y>0$ by [18, p. 144]

$$\mathfrak{R}(y,w):=\int_0^{\infty}\frac{t\cos(yt)}{t^2+w^2}\,dt.$$

Before stating the generalization of (1.18), that is sought for, we first introduce a new generalization of Raabe's cosine transform, valid for $\text{Re}(w)>0, \text{Re}(z)>0$ and $y>0$, by

$$\mathfrak{R}_z(y,w):=\frac{1}{2}\Gamma(2z+1)\int_0^{\infty}\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)\cos(yt)\,dt. \tag{1.19}$$

It is easy to see that $\mathfrak{R}_0(y,w)=\mathfrak{R}(y,w)$. Also for $w>0$, $\mathfrak{R}_z(y,w)$ satisfies a nice identity, namely,

$$w^{2z}\mathfrak{R}_z(y,w)=y^{2z}\mathfrak{R}_z(w,y), \tag{1.20}$$

which is easily seen by making the change of variable $t=xw/y$ in (1.19).

Our first result on $\mathfrak{R}_z(y,w)$ gives a closed-form evaluation of an infinite series containing $\mathfrak{R}_z(y,w)$.

Theorem 1.9. *Let $\zeta(z,a)$ be the Hurwitz zeta function. For $\text{Re}(w)>0$ and $\text{Re}(z)>0$, we have*

$$\begin{aligned}\frac{2}{\Gamma(2z+1)}\sum_{n=1}^{\infty}\mathfrak{R}_z(2\pi n,w)&=\sum_{n=1}^{\infty}\int_0^{\infty}\left(\frac{1}{(v-iw)^{2z+1}}+\frac{1}{(v+iw)^{2z+1}}\right)\cos(2\pi nv)\,dv\\&=\frac{1}{2}\left\{\zeta(1+2z,iw)+\zeta(1+2z,-iw)-\frac{\cos(\pi z)}{zw^{2z}}\right\}.\end{aligned} \tag{1.21}$$

Note that this result is not straightforward to obtain as one cannot interchange the order of the summation and integration as doing so leads to a divergent integral. The primary tool to prove this result is Guinand's generalization of Poisson's summation formula [22, Theorem 1]; see Theorem 2.2.

The generalized Raabe cosine transform $\mathfrak{R}_z(y,w)$ itself can be evaluated in terms of exponential integral functions and incomplete gamma functions which are not so popular. However, the beauty of Theorem 1.9 is that the infinite sum of $\mathfrak{R}_z(y,w)$ can be evaluated in terms of the well-known functions such as Hurwitz zeta function $\zeta(z,a)$ and $\cos(z)$.

An immediate consequence of Theorem 1.9 is

Corollary 1.10. *Equation (1.18) holds true.*

Our next result gives an evaluation of a double integral which is imperative to prove Theorem 1.9.

Theorem 1.11. *Let* $\mathrm{Re}(w) > 0$ *and* $\mathrm{Re}(z) > 0$. *Then*

$$\int_0^\infty \int_0^\infty \left(\frac{1}{(t-iw)^{2z+1}} + \frac{1}{(t+iw)^{2z+1}} \right) \cos(2\pi vt)\ dtdv = -\frac{1}{2w^{2z+1}} \sin(\pi z). \tag{1.22}$$

Equivalently, in the notation of (1.19),

$$\int_0^\infty \mathfrak{R}_z(2\pi v, w)\ dv = -\frac{1}{4w^{2z+1}} \Gamma(2z+1) \sin(\pi z).$$

It is effortless to see that for $z \in \mathbb{N} \cup \{0\}$, the above integral evaluates to zero. The particular case $z = 0$ is already obtained in [14, Lemma 3.4].

We now provide an new equivalent representation for $\mathfrak{R}_m(1, w)$, where $m \in \mathbb{N} \cup \{0\}$. This representation appears in the transformation of $\sum_{n=1}^\infty \sigma_{2m}(n) e^{-ny}$ given in Theorem 1.5.

Theorem 1.12. *Let* $\mathrm{Shi}(z)$, $\mathrm{Chi}(z)$ *and* $\mathfrak{R}_z(y, w)$ *be defined in* (1.10) *and* (1.19) *respectively. Let* $m \in \mathbb{N} \cup \{0\}$ *and* $\mathrm{Re}(w) > 0$. *Then*

$$\begin{aligned} \mathfrak{R}_m(1, w) &= \frac{(-1)^m (2m)!}{2} \int_0^\infty \left(\frac{1}{(t-iw)^{2m+1}} + \frac{1}{(t+iw)^{2m+1}} \right) \cos(t)\ dt \\ &= \sinh(w)\mathrm{Shi}(w) - \cosh(w)\mathrm{Chi}(w) + \sum_{j=1}^{m} (2j-1)! w^{-2j}. \end{aligned} \tag{1.23}$$

The special case $m = 0$ of this result was derived in [16, Lemma 9.1].

As mentioned earlier, we provide a new proof of the Theorem 1.5 in this paper. It is done by employing the Lipschitz summation formula and Theorems 1.9 and 1.12. Deriving it this way is simpler than obtaining it as a special case of (1.9). The latter was done in [16, Section 9].

This paper is organised as follows. In Section 2, the proofs of Theorems 1.9, 1.11 and 1.12 are given. Sections 3 and 4 are devoted to proving Theorems 1.1 and 1.5 respectively.

2. The generalized Raabe cosine transform $\mathfrak{R}_z(y, w)$

This section is devoted to obtaining the results associated with $\mathfrak{R}_z(y, w)$ and which are crucial to proving Theorem 1.5. The first result below gives the asymptotic expansion of $\mathfrak{R}_z(y, w)$ as $y \to \infty$.

Lemma 2.1. *Let* $\mathfrak{R}_z(y, w)$ *be defined in* (1.19). *Let* $\mathrm{Re}(w) > 0$ *and* $\mathrm{Re}(z) > 0$. *Then as* $y \to \infty$,

$$\mathfrak{R}_z(y, w) \sim -\frac{\cos(\pi z)}{w^{2z}} \sum_{n=1}^\infty \frac{\Gamma(2z+2n)}{(yw)^{2n}}.$$

Proof. We use the analogue of Watson's lemma for Laplace transform in the setting of Fourier transforms [35], [13, Equations (1.3), (1.4)]. It states that if the form of $h(t)$ near $t = 0$ is given as a series of algebraic powers, that is,

$$h(t) \sim \sum_{n=0}^\infty b_n t^{n+\lambda-1} \tag{2.1}$$

as $t \to 0^+$, then under certain restrictions on h (see [35], [13, Section 2] for the same),

$$\int_0^\infty e^{ist} h(t)\, \mathrm{d}t \sim \sum_{n=0}^\infty b_n e^{i(n+\lambda)\pi/2} \Gamma(n+\lambda) s^{-n-\lambda} \tag{2.2}$$

as $s \to \infty$.

Let

$$h(t) := \frac{1}{(t-iw)^{2z+1}} + \frac{1}{(t+iw)^{2z+1}}.$$

Then, near $t = 0$, it is easy to see that

$$\begin{aligned} h(t) &= (-iw)^{-(2z+1)} \sum_{n=0}^\infty \frac{(2z+1)_n}{n!} \left(\frac{t}{iw}\right)^n + (iw)^{-(2z+1)} \sum_{n=0}^\infty \frac{(2z+1)_n}{n!} \left(-\frac{t}{iw}\right)^n \\ &= 2w^{-(2z+1)} \sum_{n=0}^\infty \frac{(2z+1)_n}{n! w^n} \sin\left(\frac{\pi n}{2} - \pi z\right) t^n. \end{aligned}$$

Therefore, it is clear that our function $h(t)$ satisfies (2.1) with $\lambda = 1$ and

$$b(n) = \frac{2w^{-(2z+1)} (2z+1)_n}{n! w^n} \sin\left(\frac{\pi n}{2} - \pi z\right). \tag{2.3}$$

From (2.2) and (2.3), as $y \to \infty$,

$$\int_0^\infty \left(\frac{1}{(t-iw)^{2z+1}} + \frac{1}{(t+iw)^{2z+1}}\right) e^{iyt}\ dt \sim \sum_{n=0}^\infty b_n e^{i(n+1)\frac{\pi}{2}} \Gamma(n+1) y^{-n-1}, \tag{2.4}$$

where $b(n)$ is given in (2.3). Similarly, as $y \to \infty$,

$$\int_0^\infty \left(\frac{1}{(t-iw)^{2z+1}} + \frac{1}{(t+iw)^{2z+1}}\right) e^{-iyt}\ dt \sim \sum_{n=0}^\infty b_n e^{i(n+1)\frac{\pi}{2}} \Gamma(n+1) (-y)^{-n-1}. \tag{2.5}$$

From (2.4) and (2.5), we see that as $y \to \infty$,

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{(t-iw)^{2z+1}} + \frac{1}{(t+iw)^{2z+1}}\right) \cos(yt)\ dt \\ &\sim w^{-(2z+1)} \sum_{n=0}^\infty \frac{(2z+1)_n}{w^n y^{n+1}} e^{i(n+1)\frac{\pi}{2}} \sin\left(\frac{\pi n}{2} - \pi z\right) \left(1 + (-1)^{-n-1}\right) \\ &= 2w^{-(2z+1)} \sum_{n=1}^\infty \frac{(2z+1)_{(2n-1)}}{w^{2n-1} y^{2n}} e^{n\pi i} \sin\left(\frac{\pi(2n-1)}{2} - \pi z\right) \\ &= -\frac{2w^{-2z} \cos(\pi z)}{\Gamma(2z+1)} \sum_{n=1}^\infty \frac{\Gamma(2z+2n)}{(yw)^{2n}}. \end{aligned} \tag{2.6}$$

Lemma 2.1 follows upon multiplying both sides of (2.6) by $\frac{1}{2}\Gamma(2z+1)$ and then using the definition of $\mathfrak{R}_z(y, w)$ from (1.19). ◻

Remark 1. *The special case $z = 0$ of Lemma 2.1 was obtained in* [14, Lemma 3.3].

Our next task is to evaluate the double integral in (1.22).

Proof of Theorem 1.11. Note that double integral in (1.22) is not absolutely convergent which means we cannot interchange the order of integration. Securing convergence of the integral over v near $v=0$ is straightforward. Along with this, Lemma 2.1 implies that the double integral in (1.22) is convergent.

We first evaluate a more general integral by introducing the exponential factor $e^{-\frac{v^2}{N}}$ inside the integrand and then take limit $N\to\infty$. Let N be a positive integer and consider the integral

$$I(w,z,N):=\int_0^\infty\int_0^\infty e^{-\frac{v^2}{N}}\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)\cos(2\pi vt)\ dtdv\quad (\mathrm{Re}(w)>0,\ \mathrm{Re}(z)>0). \tag{2.7}$$

By invoking Fubini's theorem we can interchange the order of the summation and integration in the above equation to see that

$$\begin{aligned}I(w,z,N)&=\int_0^\infty\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)\int_0^\infty e^{-\frac{v^2}{N}}\cos(2\pi vt)\ dvdt\\&=\frac{\sqrt{\pi N}}{2}\int_0^\infty e^{-N\pi^2t^2}\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)dt,\end{aligned} \tag{2.8}$$

where we used the fact that $e^{-v^2/N}$ is self-reciprocal (up to some factor) with respect to the cosine kernel (See [21, p. 488, Formula 3.896.4]). Next invoke the identity [4, p. 88, Section 2.5.5]

$$(1-\sqrt{\xi})^{-2s}+(1+\sqrt{\xi})^{-2s}=2\ {}_2F_1\left(s,s+\frac{1}{2};\frac{1}{2};\xi\right),$$

with $\xi=-w^2/t^2$ and $s=z+1/2$ in (2.8) to deduce that

$$\begin{aligned}I(w,z,N)&=\sqrt{\pi N}\int_0^\infty e^{-N\pi^2t^2}t^{-2z-1}{}_2F_1\left(z+\frac{1}{2},z+1;\frac{1}{2};-\frac{w^2}{t^2}\right)dt\\&=\frac{\sqrt{\pi N}}{2}\int_0^\infty e^{-N\pi^2/x}x^{z-1}{}_2F_1\left(z+\frac{1}{2},z+1;\frac{1}{2};-w^2x\right)dx,\end{aligned} \tag{2.9}$$

where we made the change of variable $t=1/\sqrt{x}$. From [39, p 319, Formula 2.21.2.6], for $\mathrm{Re}(p)>0, \mathrm{Re}(a-\alpha)>0,\ \mathrm{Re}(b-\alpha)>0$ and $|\arg(\omega)|<\pi$, we have

$$\begin{aligned}\int_0^\infty x^{\alpha-1}e^{-p/x}{}_2F_1(a,b;c;-\omega x)dx=\omega^{-\alpha}\frac{\Gamma(c)\Gamma(\alpha)\Gamma(a-\alpha)\Gamma(b-\alpha)}{\Gamma(a)\Gamma(b)\Gamma(c-\alpha)}{}_2F_2(a-\alpha,b-\alpha;1-\alpha,c-\alpha;\omega p)\\+p^\alpha\Gamma(-\alpha){}_2F_2(a,b;c,\alpha+1;\omega p).\end{aligned}$$

Let $p=N\pi^2,\ a=z+1/2,\ b=z+1,\ c=1/2,\ \alpha=z$ and $\omega=w^2$ in the above integral evaluation, use the reflection formula for the gamma function $\Gamma(1/2+s)\Gamma(1/2-s)=\pi/\cos(\pi s)$ and substitute the resultant in (2.9) so that for $|\arg(w)|<\pi/2$,

$$I(w,z,N)=\frac{\sqrt{\pi N}}{2}\left\{\frac{\cos(\pi z)}{zw^{2z}}{}_2F_2\left(\frac{1}{2},1;1-z,\frac{1}{2}-z;N\pi^2w^2\right)+(N\pi^2)^z\Gamma(-z){}_1F_1\left(z+\frac{1}{2};\frac{1}{2};N\pi^2w^2\right)\right\}. \tag{2.10}$$

We now wish to take limit $N\to\infty$ on both sides of the above equation. To that end, we need to find the behavior of the functions on the right-hand side as $N\to\infty$. The following asymptotic is given by Kim [27]: as $x\to\infty$ in $-\frac{3\pi}{2}<\arg(x)<\frac{\pi}{2}$, for $\alpha\neq\mathbb{Z}\cup\{0\}$,

$${}_2F_1(1,\alpha;\rho_1,\rho_2;x)\sim\frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\alpha)}\left(K_{22}(x)+L_{22}(-x)\right),$$

where, with $\nu = 1+\alpha-\rho_1-\rho_2$,

$$K_{22}(x) = x^{\nu}e^{x}{}_2F_0\left(\rho_1-\alpha, \rho_2-\alpha; -; \frac{1}{x}\right),$$

and

$$\begin{aligned}L_{22}(x) = x^{-1}&\frac{\Gamma(\alpha-1)}{\Gamma(\rho_1-1)\Gamma(\rho_2-1)}{}_3F_1\left(1, 2-\rho_1, 2-\rho_2; 2-\alpha; \frac{1}{x}\right)\\&+x^{-\alpha}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\rho_1-\alpha)\Gamma(\rho_1-\alpha)}{}_2F_0\left(1+\alpha-\rho_1, 1+\alpha-\rho_2; -; \frac{1}{x}\right).\end{aligned}$$

We let $\alpha = 1/2$, $\rho_1 = 1-z$, $\rho_2 = 1/2-z$ and $x = N\pi^2w^2$ in the above expression to get, for $-\frac{3\pi}{4} < \arg(w) < \frac{\pi}{4}$,

$$\begin{aligned}{}_2F_2\left(\frac{1}{2}, 1; 1-z, \frac{1}{2}-z; N\pi^2w^2\right) \sim \frac{\Gamma(1/2-z)\Gamma(1-z)}{\sqrt{\pi}}&\left\{(N\pi^2w^2)^{2z}e^{N\pi^2w^2}{}_2F_0\left(\frac{1}{2}-z, -z; -; \frac{1}{N\pi^2w^2}\right)\right.\\&+\frac{2}{N\pi^{3/2}w^2\Gamma(-1/2-z)\Gamma(-z)}{}_3F_1\left(1, 1+z, \frac{3}{2}+z; \frac{3}{2}; -\frac{1}{N\pi^2w^2}\right)\\&\left.+\frac{\pi(-N\pi^2w^2)^{-1/2}}{\Gamma(1/2-z)\Gamma(-z)}{}_2F_0\left(\frac{1}{2}+z, 1+z; -; -\frac{1}{N\pi^2w^2}\right)\right\}\end{aligned}\tag{2.11}$$

as $N\to\infty$. Also, from [43, p. 189, Exercise 7.7],

$${}_1F_1(a;c;x) \sim \frac{e^x x^{a-c}\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(c-a)_n(1-a)_n}{n!}x^{-n} + \frac{e^{-\pi i a}x^{-a}}{\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{(a)_n(1+a-c)_n}{n!}(-x)^{-n}, \quad x\to\infty,$$

where $-\frac{3\pi}{2} < \arg(x) < \frac{\pi}{2}$. Upon letting $a = z+1/2$, $c = 1/2$ and $x = N\pi^2w^2$ in the above formula and using the series definition of ${}_2F_0$, for $-\frac{3\pi}{4} < \arg(w) < \frac{\pi}{4}$, we see that

$$\begin{aligned}{}_1F_1\left(z+\frac{1}{2}; \frac{1}{2}; N\pi^2w^2\right) \sim e^{N\pi^2w^2}&\frac{\sqrt{\pi}(N\pi^2w^2)^z}{\Gamma(z+1/2)}{}_2F_0\left(-z, \frac{1}{2}-z; -; \frac{1}{N\pi^2w^2}\right)\\&+e^{-\pi i(z+1/2)}\frac{\sqrt{\pi}(N\pi^2w^2)^{-(z+1/2)}}{\Gamma(-z)}{}_2F_0\left(z+\frac{1}{2}, 1+z; -; -\frac{1}{N\pi^2w^2}\right)\end{aligned}\tag{2.12}$$

as $N\to\infty$. Substitute (2.11) and (2.12) in (2.10) and observe that the terms involving ${}_2F_0\left(-z, \frac{1}{2}-z; -; \frac{1}{N\pi^2w^2}\right)$ cancel each other out. Also note that ${}_pF_q(a_1,\cdots,a_p; b_1,\cdots,b_q; 1/N) = 1+O(1/N)$, as $N\to\infty$. Hence, for $-\frac{\pi}{2} < \arg(w) < \frac{\pi}{4}$, as $N\to\infty$,

$$\begin{aligned}I(w,z,N) = \frac{\sqrt{\pi}}{2}&\left\{\frac{2^{2z}\Gamma(1-2z)\cos(\pi z)}{zw^{2z}}\left[\frac{1}{\sqrt{N}}\frac{2^{-2z-1}}{\pi^2w^2\Gamma(-1-2z)}\left(1+O\left(\frac{1}{N}\right)\right)\right.\right.\\&\left.\left.+\frac{i2^{-2z-1}}{\sqrt{\pi}w\Gamma(-2z)}\left(1+O\left(\frac{1}{N}\right)\right)\right]-i\frac{1}{\sqrt{\pi}}e^{-\pi iz}w^{-2z-1}\left(1+O\left(\frac{1}{N}\right)\right)\right\}.\end{aligned}$$

We next let $N\to\infty$ on the both sides of the above equation. By using the dominated convergence theorem, we can take the limit $N\to\infty$ inside the integral sign in (2.7). Thus,

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty}\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)\cos(2\pi vt)\ dtdv &= -\frac{i}{2w^{2z+1}}\left\{e^{-\pi iz}-\cos(\pi z)\right\}\\&= -\frac{i}{2w^{2z+1}}\left\{e^{-\pi iz}-\frac{e^{i\pi z}+e^{-i\pi z}}{2}\right\}\end{aligned}$$

$$= \frac{i}{2w^{2z+1}}\left\{\frac{e^{i\pi z}-e^{-i\pi z}}{2}\right\}, \tag{2.13}$$

which proves our theorem for $-\frac{\pi}{2} < \arg(w) < \frac{\pi}{4}$. We next prove the result in the remaining region $\frac{\pi}{4} \le \arg(w) < \frac{\pi}{2}$.

By invoking the asymptotic [36, p. 411, Formula 16.11.7] twice, for $\frac{\pi}{4} \le \arg(w) < \frac{\pi}{2}$, as $N \to \infty$,

$$\begin{aligned} {}_2F_2\left(\frac{1}{2}, 1; 1-z, \frac{1}{2}-z; N\pi^2 w^2\right) \sim \frac{\Gamma(1-z)\Gamma\left(\frac{1}{2}-z\right)}{\Gamma(1/2)} &\left\{\frac{e^{\pi i/2}}{\sqrt{N}\pi w}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\frac{1}{2}+k\right)\Gamma\left(\frac{1}{2}-k\right)}{k!\Gamma\left(\frac{1}{2}-z-k\right)\Gamma\left(-z-k\right)}\left(N\pi^2w^2e^{-\pi i}\right)^{-k}\right. \\ &+\frac{e^{\pi i}}{N\pi^2w^2}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(-\frac{1}{2}-k\right)}{k!\Gamma\left(-z-k\right)\Gamma\left(-\frac{1}{2}-z-k\right)}\left(N\pi^2w^2e^{-\pi i}\right)^{-k} \\ &\left.+\left(N\pi^2w^2\right)^{2z}e^{N\pi^2w^2}\sum_{k=0}^{\infty}C_k(N\pi^2w^2)^{-k}\right\}, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} {}_1F_1\left(z+\frac{1}{2}; \frac{1}{2}; N\pi^2w^2\right) \sim \frac{\Gamma(1/2)}{\Gamma\left(z+\frac{1}{2}\right)} &\left\{(N\pi^2w^2e^{-\pi i})^{-(z+1/2)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\frac{1}{2}+z+k\right)}{k!\Gamma(k-z)}(N\pi^2w^2e^{-\pi i})^{-k}\right. \\ &\left.+(N\pi^2w^2)^z e^{N\pi^2w^2}\sum_{k=0}^{\infty}C_k(N\pi^2w^2)^{-k}\right\}, \end{aligned} \tag{2.15}$$

where

$$C_k = -\frac{1}{k}\sum_{m=0}^{k-1}C_m e_{k,m},$$

with $C_0 = 1$ and

$$e_{k,m} = 2(m-z)_{(k+1-m)}\left(z-\frac{1}{2}\right) - 2z\left(\frac{1}{2}-z+m\right)_{(k+1-m)}.$$

Upon simplifying (2.9), (2.14) and (2.15), and observing that the terms containing $e^{N\pi^2w^2}$ cancel each other out, for $\frac{\pi}{4} \le \arg(w) < \frac{\pi}{2}$,

$$I(w,z,N) = \frac{e^{\pi i/2}}{2w^{2z+1}}\left(e^{\pi i z} - \cos(\pi z)\right) + O\left(\frac{1}{\sqrt{N}}\right)$$

as $N \to \infty$. Employing the dominated convergence theorem to take limit $N \to \infty$ inside the double integral, we deduce that

$$\int_0^\infty\int_0^\infty\left(\frac{1}{(t-iw)^{2z+1}} + \frac{1}{(t+iw)^{2z+1}}\right)\cos(2\pi vt)\ dtdv = -\frac{1}{2w^{2z+1}}\sin(\pi z).$$

This along with (2.13) completes the proof of the theorem for $-\frac{\pi}{2} < \arg(w) < \frac{\pi}{2}$. □

As discussed in the introduction, Guinand's generalization of Poisson's summation formula [22, Theorem 1] is critical to prove Theorem 1.9. We record Guinand's result in the following theorem.

Theorem 2.2. *If $f(x)$ is an integral, $f(x)$ tends to zero as $x \to \infty$, and $xf'(x)$ belongs to $L^p(0,\infty)$, for some p, $1 < p \le 2$, then*

$$\lim_{M\to\infty}\left(\sum_{m=1}^{M} f(m) - \int_0^M f(v)\,\mathrm{d}v\right) = \lim_{M\to\infty}\left(\sum_{m=1}^{M} g(m) - \int_0^M g(v)\,\mathrm{d}v\right),$$

where

$$g(x)=2\int_0^{\to\infty} f(t)\cos(2\pi xt)\,\mathrm{d}t.$$

Proof of Theorem 1.9. Let

$$f(v):=\frac{1}{(v-iw)^{2z+1}}+\frac{1}{(v+iw)^{2z+1}},\quad (\mathrm{Re}(w)>0,\mathrm{Re}(z)>0)\,,$$

$$g(x):=2\int_0^{\infty}\left(\frac{1}{(v-iw)^{2z+1}}+\frac{1}{(v+iw)^{2z+1}}\right)\cos(2\pi xv)\ dv. \tag{2.16}$$

Now employ Theorem 2.2 with $f(x)$ and $g(x)$ as above. Invoking Theorem 1.11, we see that

$$\sum_{n=1}^{\infty}g(n)=\lim_{M\to\infty}\left\{\sum_{n=1}^{M}\left(\frac{1}{(n-iw)^{2z+1}}+\frac{1}{(n+iw)^{2z+1}}\right)\right.$$
$$\left.-\int_0^{M}\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)\ dt\right\}-\frac{1}{w^{2z+1}}\sin(\pi z). \tag{2.17}$$

Note that series and integral on the right-hand side of the above equation exist individually in the limit $M\to\infty$. Therefore,

$$\sum_{n=1}^{\infty}g(n)=\sum_{n=1}^{\infty}\left(\frac{1}{(n-iw)^{2z+1}}+\frac{1}{(n+iw)^{2z+1}}\right)-\int_0^{\infty}\left(\frac{1}{(t-iw)^{2z+1}}+\frac{1}{(t+iw)^{2z+1}}\right)\ dt-\frac{1}{w^{2z+1}}\sin(\pi z). \tag{2.18}$$

It is easy to see that for $\mathrm{Re}(z)>0$,

$$\sum_{n=1}^{\infty}\frac{1}{(n\mp iw)^{2z+1}}=\zeta(1+2z,1\mp iw). \tag{2.19}$$

Also,

$$\int_0^{\infty}\frac{dt}{(t\mp iw)^{2z+1}}=\frac{(\mp i)^{-2z}w^{-2z}}{2z}. \tag{2.20}$$

Substitute (2.19) and (2.20) in (2.18) to deduce that

$$\sum_{n=1}^{\infty}g(n)=\zeta(1+2z,iw)+\zeta(1+2z,-iw)-\frac{\cos(\pi z)}{zw^{2z}}-\frac{1}{w^{2z+1}}\sin(\pi z)$$
$$=\zeta(1+2z,iw)+\zeta(1+2z,-iw)-\frac{\cos(\pi z)}{zw^{2z}}, \tag{2.21}$$

which follows using the fact

$$\zeta(s,a+1)=\zeta(s,a)-a^{-s}. \tag{2.22}$$

Therefore, (2.16) and (2.21) yield Theorem 1.9. □

Proof of Corollary 1.10. We wish to take limit $z\to 0$ in (1.21). To that end, we use expansions of the functions involved around $z=0$. As $s\to 1$, we have [21, p. 1038, Formula 9.533.2]

$$\zeta(s,a)=\frac{1}{s-1}-\psi(a)+O(|s-1|).$$

The above equation implies that, as $z\to 0$,

$$\zeta(1+2z,\pm iw)=\frac{1}{2z}-\psi(\pm iw)+O(|z|). \tag{2.23}$$

It is easy to see that

$$\frac{\cos(\pi z)}{zw^{2z}} = \frac{1}{z} - 2\log(w) + O(|z|), \tag{2.24}$$

as $z \to 0$. Using (2.23) and (2.24), we deduce that

$$\lim_{z\to 0}\left(\zeta(1+2z, iw) + \zeta(1+2z, -iw) - \frac{\cos(\pi z)}{zw^{2z}}\right) = -\psi(iw) - \psi(-iw) + 2\log(w). \tag{2.25}$$

Let $z \to 0$ on both sides of (1.21) and use (2.25) so that

$$2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{v\cos(2\pi nv)}{v^2+w^2}\,dv = \frac{1}{2}\left\{2\log(w) - (\psi(iw) + \psi(-iw))\right\}. \tag{2.26}$$

Make the change of variable $2\pi nv = t$ on the right-hand side of (2.26) to arrive at

$$2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{t\cos(t)}{t^2+(2\pi w)^2n^2}\,dt = \log(w) - \frac{1}{2}\left(\psi(iw) + \psi(-iw)\right).$$

Finally let $w = u/(2\pi)$ in the above equation to conclude the proof of the corollary. □

Theorem 1.12 is proved next.

Proof of Theorem 1.12. From [16, Lemma 9.1], for $\mathrm{Re}(w) > 0$, we have

$$\int_0^{\infty}\frac{t\cos t\,dt}{t^2+w^2} = \sinh(w)\mathrm{Shi}(w) - \cosh(w)\mathrm{Chi}(w). \tag{2.27}$$

Now (1.23) follows by expanding $\frac{t}{(t^2+w^2)}$ in partial fractions, that is, by writing $\frac{t}{t^2+w^2} = \frac{1}{2}\left(\frac{1}{t-iw} + \frac{1}{t+iw}\right)$, and then by performing integration by parts $2m$ times the left-hand side of (2.27). □

3. Proof of our generalization of a formula of Ramanujan

We first find an inverse Mellin transform which will be used to prove Theorem 1.1.

Lemma 3.1. *For* $1 < d := \mathrm{Re}(z) < 3$ *and* $x \notin \mathbb{Z}$,

$$\cot(\pi x) = \frac{1}{2\pi i}\int_{(d)}\zeta(1-z)\tan\left(\frac{\pi z}{2}\right)x^{-z}dz, \tag{3.1}$$

where, here, and in the sequel, $\int_{(c)} dz$ *represents the line integral* $\int_{c-i\infty}^{c-i\infty} dz$ *with* $c = \mathrm{Re}(z)$.

Proof. We want to use the series representation of $\zeta(1-z)$ to evaluate the integral on the right-hand side of (3.1). To that end, we shift the line of integration to $-1 < c < 0$, use residue theorem while noting that the integrals along the horizontal segments tend to zero as the height of the contour tends to ∞, to deduce that

$$\begin{aligned}\frac{1}{2\pi i}\int_{(d)}\zeta(1-z)\tan\left(\frac{\pi z}{2}\right)x^{-z}dz &= \frac{1}{\pi x} + \frac{1}{2\pi i}\int_{(c)}\zeta(1-z)\tan\left(\frac{\pi z}{2}\right)x^{-z}dz\\ &= \frac{1}{\pi x} + \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{2\pi i}\int_{(c)}\tan\left(\frac{\pi z}{2}\right)\left(\frac{x}{n}\right)^{-z}dz. \end{aligned} \tag{3.2}$$

We employ [34, p. 182, Formula 2.4.4],

$$\frac{1}{2\pi i}\int_{(c_1)}\tan\left(\frac{\pi z}{2}\right)y^{-z}dz = \frac{2}{\pi}\frac{y}{y^2-1}, \qquad (-1 < c_1 < 1, y \neq \pm 1),$$

with y replaced by x/n in (3.2) so that for $x \notin \mathbb{Z}$,

$$\frac{1}{2\pi i}\int_{(d)}\zeta(1-z)\tan\left(\frac{\pi z}{2}\right)x^{-z}dz=\frac{1}{\pi x}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2x}{x^2-n^2}.$$

Equation (3.1) now follows upon using the well-known fact

$$\pi\cot(\pi x)=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2x}{x^2-n^2}. \tag{3.3}$$

□

Next, we note the following result of Hardy [26, pp. 56-57]. This result helps us justify the interchange of the order of the summation and integration having principal values, and will be employed in the proof of Theorem 1.1.

Proposition 3.2. *Let*

$$S(x)=\sum_{k=0}^{\infty}u_k(x)$$

be a series whose terms are functions of x and is convergent with the possible exception of a closed enumerable set of points for values of x in a finite interval (a, A). Let α denote one such point in this set. If

(1) *the series $S(x)$ is integrable term by term over any part of (a, A) which does not include α,*
(2) *the function*

$$F(x)=\sum_{k=0}^{\infty}\mathrm{PV}\int_{a}^{x}u_k(t)dt$$

is a continuous function of x except at α, and

(3)

$$\lim_{\epsilon\to0}\{F(\alpha-\epsilon)-F(\alpha+\epsilon)\}=0. \tag{3.4}$$

Then, one can interchange the order of summation and integration, namely,

$$\mathrm{PV}\int_{a}^{A}\sum_{k=0}^{\infty}u_k(t)dt=\sum_{k=0}^{\infty}\mathrm{PV}\int_{a}^{A}u_k(t).$$

Remark 2. *We note that* [26, pp. 58–59, Section 7] (*also see* [25, p. 27]) *if*

$$u_k(x)=\frac{v_k(x)}{x-\alpha}, \tag{3.5}$$

where $v_k(x)$ is a function of x and has a continuous derivative for all $x\in[a, A]$, then

$$\mathrm{PV}\int_{\alpha-\epsilon}^{\alpha+\epsilon}u_k(x)dx=2\epsilon v_k'(\alpha+\mu),\ \text{for some } \mu\in[-\epsilon,\epsilon].$$

Also if $|v_k'(x)|<V_k$ for all values $x\in[a, A]$, V_k being independent of x and $\sum_{k=0}^{\infty}V_k$ is convergent, then the condition (3.4) *holds true for $u_k(x)$ given in* (3.5)*.*

In the next lemma, we justify the interchange of the order of the summation and principal value integral.

Lemma 3.3. *Let* $k \in \mathbb{N}$, $0 < a \leq 1$ *and* $\mathrm{Re}(y) > 0$. *For* $\mathrm{Re}(s) > 2$, *we have*

$$\sum_{k=1}^{\infty} \sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty} x^{s-1}e^{-4\pi^2 kx/y}\cot(\pi x)dx = \mathrm{PV}\int_0^{\infty}\left(\sum_{k=1}^{\infty}\sin(2\pi ak)\,e^{-4\pi^2 kx/y}\right)x^{s-1}\cot(\pi x)dx. \tag{3.6}$$

Proof. Note that the presence of $\cot(\pi x)$ implies infinitely many singularities of the integrand on the left side of (3.6). To handle this integral efficiently, we use (3.3) so that

$$\begin{aligned}\sum_{k=1}^{\infty} \sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty} x^{s-1}e^{-4\pi^2 kx/y}\cot(\pi x)dx &= \frac{1}{\pi}\sum_{k=1}^{\infty}\sin(2\pi ak)\int_0^{\infty} x^{s-2}e^{-4\pi^2 kx/y}dx\\ &\quad + \frac{2}{\pi}\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty} x^{s}e^{-4\pi^2 kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx.\end{aligned} \tag{3.7}$$

We can interchange the order of summation and integration in the first expression on the right-hand side of (3.7) by easily employing [43, p. 30, Theorem 2.1]. The delicate part is to show the same for the second expression on the right, which is done next. We first show that

$$\mathrm{PV}\int_0^{\infty} x^{s}e^{-4\pi^2 kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx = \sum_{n=1}^{\infty}\mathrm{PV}\int_0^{\infty}\frac{x^{s}e^{-4\pi^2 kx/y}}{x^2-n^2}dx. \tag{3.8}$$

The ingenious argument given in [10, pp. 909-911] can be adapted here as well to prove the above claim. We give the complete details though to make the paper self-contained.

Let $w(t) \in C_0^{\infty}$ be a smooth function such that $0 \leq w(t) \leq 1$, $\forall\, t \in \mathbb{R}$, $w(t)$ has compact support in $\left(-\frac{1}{3}, \frac{1}{3}\right)$, and $w(t) = 1,\ t \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. Note that the right-hand side of (3.8) can be rewritten as

$$\sum_{n=1}^{\infty}\mathrm{PV}\int_0^{\infty}\frac{x^{s}e^{-4\pi^2 kx/y}}{x^2-n^2}dx = \sum_{n=1}^{\infty}\int_0^{\infty} x^{s}e^{-4\pi^2 kx/y}\frac{(1-w(x-n))}{x^2-n^2}dx + \sum_{n=1}^{\infty}\mathrm{PV}\int_0^{\infty} x^{s}e^{-4\pi^2 kx/y}\frac{w(x-n)}{x^2-n^2}dx. \tag{3.9}$$

Again, an easy application of [43, p. 30, Theorem 2.1] allows us to interchange the order of summation and integration in the first expression of (3.9). If m is a positive integer such that $m - \frac{1}{2} \leq x \leq m + \frac{1}{2}$, then

$$\sum_{n=1}^{\infty}\frac{w(x-n)}{x^2-n^2} = \frac{w(x-m)}{x^2-m^2}. \tag{3.10}$$

Hence, using (3.10) in the second step below, we have

$$\begin{aligned}\mathrm{PV}\int_0^{\infty} x^{s}e^{-4\pi^2 kx/y}\sum_{n=1}^{\infty}\frac{w(x-n)}{x^2-n^2}dx &= \sum_{m=1}^{\infty}\mathrm{PV}\int_{m-1/2}^{m+1/2} x^{s}e^{-4\pi^2 kx/y}\sum_{n=1}^{\infty}\frac{w(x-n)}{x^2-n^2}dx\\ &= \sum_{m=1}^{\infty}\mathrm{PV}\int_{m-1/2}^{m+1/2} x^{s}e^{-4\pi^2 kx/y}\frac{w(x-m)}{x^2-m^2}dx\\ &= \sum_{m=1}^{\infty}\mathrm{PV}\int_0^{\infty} x^{s}e^{-4\pi^2 kx/y}\frac{w(x-m)}{x^2-m^2}dx.\end{aligned}$$

The above fact along with (3.9) gives

$$\sum_{n=1}^{\infty}\mathrm{PV}\int_0^{\infty}\frac{x^se^{-4\pi^2kx/y}}{x^2-n^2}dx=\mathrm{PV}\int_0^{\infty}x^se^{-4\pi^2kx/y}\sum_{n=1}^{\infty}\frac{(1-w(x-n))}{x^2-n^2}dx+\mathrm{PV}\int_0^{\infty}x^se^{-4\pi^2kx/y}\sum_{n=1}^{\infty}\frac{w(x-n)}{x^2-n^2}dx$$
$$=\mathrm{PV}\int_0^{\infty}x^se^{-4\pi^2kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx.$$

This proves the claim in (3.8). Therefore, we can write

$$\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}x^se^{-4\pi^2kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx=\sum_{k=1}^{\infty}\sin(2\pi ak)\sum_{n=1}^{\infty}\mathrm{PV}\int_0^{\infty}\frac{x^se^{-4\pi^2kx/y}}{x^2-n^2}dx.$$

Fubini's theorem allows us to interchange the order of the double sum on the right-hand side of the above expression so as to obtain

$$\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}x^se^{-4\pi^2kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}\frac{x^se^{-4\pi^2kx/y}}{x^2-n^2}dx. \tag{3.11}$$

Now

$$\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}\frac{x^se^{-4\pi^2kx/y}}{x^2-n^2}dx=\sum_{k=1}^{\infty}\sin(2\pi ak)\left\{\left(\int_0^{\delta}+\int_{n+1}^{\infty}\right)\frac{x^se^{-4\pi^2kx/y}}{x^2-n^2}dx\right.$$
$$\left.+\frac{1}{2}\int_{\delta}^{n+1}\frac{x^{s-1}e^{-4\pi^2kx/y}}{x+n}dx+\frac{1}{2}\mathrm{PV}\int_{\delta}^{n+1}\frac{x^{s-1}e^{-4\pi^2kx/y}}{x-n}dx\right\}, \tag{3.12}$$

where $0<\delta<1$. Note that there is no need to take principal value for the first three integrals on the right-hand side of (3.12). Therefore, it is easy to take the summation inside these integrals using the standard techniques, for example, [43, p. 30, theorem 2.1]. To interchange the order of summation and the last integral in (3.12), we now show that the hypotheses of Proposition 3.2 are satisfied. Let us define

$$u_k(x):=\frac{v_k(x)}{x-n}\quad\text{and}\quad v_k(x):=x^{s-1}e^{-4\pi^2kx/y}\sin(2\pi ak). \tag{3.13}$$

It is easy to see that the conditions (1) and (2) of Proposition 3.2 are satisfied with $u_k(x)$ being defined in (3.13). To fulfill (3.4), we show that the equivalent condition discussed in Remark 2 is satisfied. To that end, observe that $x\in[\delta,n+1]$ and use $e^{-x}>3!/x^3, x>0$, so that

$$|v_k'(x)|<\left|x^{s-2}e^{-4\pi^2kx/y}\left(s-1-\frac{4\pi^2kx}{y}\right)\right|<\frac{x^{\mathrm{Re}(s)-5}}{(4\pi^2/y)^3}\frac{3!}{k^3}\left(|s-1|+\frac{4\pi^2kx}{y}\right)$$
$$<\frac{M}{(4\pi^2/y)^3}\frac{3!}{k^3}\left(|s-1|+\frac{4\pi^2k(n+1)}{y}\right)$$
$$=:V_k,$$

where we used the fact that the function $x^{\mathrm{Re}(s)-5}$ is continuous on the compact interval $[\delta,n+1]$, and hence bounded by some constant $M>0$ (which may depend on δ and n). Since the series $\sum_{k=1}^{\infty}V_k$ converges, all conditions of Proposition 3.2 are satisfied. Hence we can interchange the order of summation and integration even in the case of the last integral of (3.12). This fact along with the discussion following (3.12) implies that

$$\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}\frac{x^s e^{-4\pi^2 kx/y}}{x^2-n^2}dx=\mathrm{PV}\int_0^{\infty}\left(\sum_{k=1}^{\infty}\sin(2\pi ak)\,e^{-4\pi^2 kx/y}\right)\frac{x^s}{x^2-n^2}dx. \tag{3.14}$$

Using the fact $\sin(\theta)=(e^{i\theta}-e^{-i\theta})/(2i)$, we find

$$\begin{aligned}\sum_{k=1}^{\infty}\sin(2\pi ak)\,e^{-4\pi^2 xk/y}&=\frac{1}{2i}\sum_{k=1}^{\infty}e^{-\left(\frac{4\pi^2 x}{y}-2\pi ia\right)k}-\frac{1}{2i}\sum_{k=1}^{\infty}e^{-\left(\frac{4\pi^2 x}{y}+2\pi ia\right)k}\\&=\frac{1}{2i}\left(\frac{1}{e^{\frac{4\pi^2 x}{y}-2\pi ia}-1}-\frac{1}{e^{\frac{4\pi^2 x}{y}+2\pi ia}-1}\right).\end{aligned} \tag{3.15}$$

Substitute the above value in (3.14) to arrive at

$$\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}\frac{x^s e^{-4\pi^2 kx/y}}{x^2-n^2}dx=\frac{1}{2i}\mathrm{PV}\int_0^{\infty}\left(\frac{1}{e^{\frac{4\pi^2 x}{y}-2\pi ia}-1}-\frac{1}{e^{\frac{4\pi^2 x}{y}+2\pi ia}-1}\right)\frac{x^s}{x^2-n^2}dx. \tag{3.16}$$

Equations (3.11) and (3.16) yield

$$\begin{aligned}&\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}x^s e^{-4\pi^2 kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx\\&=\frac{1}{2i}\sum_{n=1}^{\infty}\mathrm{PV}\int_0^{\infty}\left(\frac{1}{e^{\frac{4\pi^2 x}{y}-2\pi ia}-1}-\frac{1}{e^{\frac{4\pi^2 x}{y}+2\pi ia}-1}\right)\frac{x^s}{x^2-n^2}dx.\end{aligned}$$

Again employing the trick that we used after (3.8) to interchange the order of the summation and integration, one can take the sum over n inside the integral on the right-hand side of the above equation to deduce that

$$\begin{aligned}&\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}x^s e^{-4\pi^2 kx/y}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}dx\\&=\frac{1}{2i}\mathrm{PV}\int_0^{\infty}\left(\frac{1}{e^{\frac{4\pi^2 x}{y}-2\pi ia}-1}-\frac{1}{e^{\frac{4\pi^2 x}{y}+2\pi ia}-1}\right)\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}x^s dx.\end{aligned} \tag{3.17}$$

Substituting (3.17) in (3.7), we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\sin(2\pi ak)\,\mathrm{PV}\int_0^{\infty}x^{s-1}e^{-4\pi^2 kx/y}\cot(\pi x)dx\\&=\frac{1}{\pi}\int_0^{\infty}x^{s-2}\sum_{k=1}^{\infty}\sin(2\pi ak)\,e^{-4\pi^2 kx/y}dx+\frac{1}{i\pi}\mathrm{PV}\int_0^{\infty}\left(\frac{1}{e^{\frac{4\pi^2 x}{y}-2\pi ia}-1}-\frac{1}{e^{\frac{4\pi^2 x}{y}+2\pi ia}-1}\right)\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}x^s dx\\&=\frac{1}{\pi}\int_0^{\infty}\left(\sum_{k=1}^{\infty}\sin(2\pi ak)\,e^{-4\pi^2 kx/y}\right)x^{s-2}dx+\frac{2}{\pi}\int_0^{\infty}\left(\sum_{k=1}^{\infty}\sin(2\pi ak)\,e^{-4\pi^2 kx/y}\right)\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}x^s dx,\end{aligned}$$

where in the ultimate step we again used (3.15). Finally employing (3.3) in the above equation, we arrive at (3.6). □

We have now collected all ingredients to give a proof of our generalization of Ramanujan's formula.

Proof of Theorem 1.1. Letting $\tau = iyj/(2\pi)$, $\mathrm{Re}(y)>0$, in Theorem 1.8, then taking summation over $j\geq 1$, and then employing the series definition of the Hurwitz zeta function for $\mathrm{Re}(s)>1$, we obtain[2]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(n-a)^{s-1}}{e^{(n-a)y}-1}&=\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{k\in\mathbb{Z}}e^{2\pi iak}\sum_{j=1}^{\infty}\frac{1}{\left(k+\frac{ijy}{2\pi}\right)^s}\\&=\frac{\Gamma(s)}{y^s}\sum_{k\in\mathbb{Z}}e^{2\pi iak}\zeta\left(s,1-\frac{2\pi ik}{y}\right)\\&=\frac{\Gamma(s)\zeta(s)}{y^s}+\frac{\Gamma(s)}{y^s}\sum_{k=1}^{\infty}\left\{e^{2\pi iak}\zeta\left(s,1-\frac{2\pi ik}{y}\right)+e^{-2\pi iak}\zeta\left(s,1+\frac{2\pi ik}{y}\right)\right\}.\end{aligned}\tag{3.18}$$

Invoking the well-known formula [36, p. 609, Formula 25.11.25]

$$\Gamma(z)\zeta(z,a)=\int_0^{\infty}\frac{e^{-ax}}{1-e^{-x}}x^{z-1}dx,\qquad (\mathrm{Re}(z)>1,\ \mathrm{Re}(a)>0)\tag{3.19}$$

in (3.18), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(n-a)^{s-1}}{e^{(n-a)y}-1}&=\frac{\Gamma(s)\zeta(s)}{y^s}+\frac{1}{y^s}\sum_{k=1}^{\infty}\int_0^{\infty}\left(e^{i\left(2\pi ak+\frac{2\pi kt}{y}\right)}+e^{-i\left(2\pi ak+\frac{2\pi kt}{y}\right)}\right)\frac{t^{s-1}}{e^t-1}dt\\&=\frac{\Gamma(s)\zeta(s)}{y^s}+\frac{2}{y^s}\sum_{k=1}^{\infty}\int_0^{\infty}\cos\left(2\pi ak+\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt\\&=\frac{\Gamma(s)\zeta(s)}{y^s}+\frac{2}{y^s}\sum_{k=1}^{\infty}\cos(2\pi ak)\int_0^{\infty}\cos\left(\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt\\&\quad-\frac{2}{y^s}\sum_{k=1}^{\infty}\sin(2\pi ak)\int_0^{\infty}\sin\left(\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt.\end{aligned}\tag{3.20}$$

Our next goal is to evaluate the integrals in (3.20). From [34, p. 42, Formula 1.5.2], for $0<\mathrm{Re}(z)<1$, we have

$$\int_0^{\infty}\cos(x)x^{z-1}dx=\Gamma(z)\cos\left(\frac{\pi z}{2}\right).$$

Making the change of variable $x=\frac{2\pi kt}{y}$ and replacing z by $s-1+z$ in the above result, we get, for $1-\mathrm{Re}(s)<\mathrm{Re}(z)<2-\mathrm{Re}(s)$,

$$\int_0^{\infty}\cos\left(\frac{2\pi kt}{y}\right)t^{s-1}t^{z-1}dt=\left(\frac{2\pi k}{y}\right)^{1-s-z}\Gamma(s-1+z)\sin\left(\frac{\pi}{2}(s+z)\right).\tag{3.21}$$

Equation (3.19) with $a=1$, (3.21), and an application of Parseval's formula [38, p. 83, Equation (3.1.14)] gives, for $1-\mathrm{Re}(s)<c=\mathrm{Re}(z)<\min(0,2-\mathrm{Re}(s))$,

$$\begin{aligned}\int_0^{\infty}\cos\left(\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt&=\left(\frac{2\pi k}{y}\right)^{1-s}\frac{1}{2\pi i}\int_{(c)}\Gamma(s-1+z)\sin\left(\frac{\pi}{2}(s+z)\right)\Gamma(1-z)\zeta(1-z)\left(\frac{2\pi k}{y}\right)^{-z}dz\\&=\left(\frac{2\pi k}{y}\right)^{1-s}\left\{\cos\left(\frac{\pi s}{2}\right)I_1(y,s)+\sin\left(\frac{\pi s}{2}\right)I_2(y,s)\right\},\end{aligned}\tag{3.22}$$

[2]The case $a=0$ of (3.18) reduces to a result of Kuylenstierna [30, Equation (7)].

where

$$I_1(y,s):=\frac{1}{2\pi i}\int_{(c)}\Gamma(s-1+z)\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z)\left(\frac{2\pi k}{y}\right)^{-z}dz, \tag{3.23}$$

$$I_2(y,s):=\frac{1}{2\pi i}\int_{(c)}\Gamma(s-1+z)\cos\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z)\left(\frac{2\pi k}{y}\right)^{-z}dz. \tag{3.24}$$

Similarly, using the formula [34, p. 42, Formula 1.5.1]

$$\int_0^\infty \sin(x)x^{z-1}dx=\Gamma(z)\sin\left(\frac{\pi z}{2}\right),\quad (-1<\mathrm{Re}(z)<1),$$

it can be seen that for $-\mathrm{Re}(s)<c=\mathrm{Re}(z)<\min(0,2-\mathrm{Re}(s))$,

$$\int_0^\infty \sin\left(\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt=\left(\frac{2\pi k}{y}\right)^{1-s}\left\{\sin\left(\frac{\pi s}{2}\right)I_1(y,s)-\cos\left(\frac{\pi s}{2}\right)I_2(y,s)\right\}. \tag{3.25}$$

We first evaluate $I_1(y,s)$. Apply the functional equation of the Riemann zeta function [36, p. 603, Formula 25.4.2]

$$\zeta(s)=2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right), \tag{3.26}$$

in (3.23) to see that

$$I_1(y,s)=\frac{\pi}{2\pi i}\int_{(c)}\Gamma(s-1+z)\zeta(z)\left(\frac{4\pi^2k}{y}\right)^{-z}dz.$$

We want to use the series definition of $\zeta(z)$ to further simplify the above integral. Therefore we shift the line of integration to $d=\mathrm{Re}(z)>1$ and use residue theorem thereby obtaining

$$\begin{aligned}I_1(y,s)&=\frac{\pi}{2\pi i}\int_{(d)}\Gamma(s-1+z)\zeta(z)\left(\frac{4\pi^2k}{y}\right)^{-z}dz-\frac{y\Gamma(s)}{4\pi k}\\&=\pi\sum_{n=1}^\infty\frac{1}{2\pi i}\int_{(d)}\Gamma(s-1+z)\left(\frac{4\pi^2nk}{y}\right)^{-z}dz-\frac{y\Gamma(s)}{4\pi k}\\&=\pi\left(\frac{4\pi^2k}{y}\right)^{s-1}\sum_{n=1}^\infty n^{s-1}e^{-\frac{4\pi^2nk}{y}}-\frac{y\Gamma(s)}{4\pi k},\end{aligned} \tag{3.27}$$

where in the last step, we used

$$e^{-x}=\frac{1}{2\pi i}\int_{(\lambda)}\Gamma(z)x^{-z}dz\qquad(\lambda>0). \tag{3.28}$$

We now focus on evaluating the other integral $I_2(y,s)$. Again an application of (3.26) in (3.24) yields

$$I_2(y,s)=\frac{\pi}{2\pi i}\int_{(c)}\Gamma(s-1+z)\zeta(z)\cot\left(\frac{\pi z}{2}\right)\left(\frac{4\pi^2k}{y}\right)^{-z}dz. \tag{3.29}$$

If we shift the line of integration from $\mathrm{Re}(z)=c$, where $1-\mathrm{Re}(s)<c<\min(0,2-\mathrm{Re}(s))$, to $1-\mathrm{Re}(s)<\mathrm{Re}(z)=c'<2$, we do not encounter any poles of the integrand in the integral of (3.29). (Note that the pole of $\zeta(z)$ at $z=1$ is annihilated by the zero of $\cot(\pi z/2)$ at $z=1$.) Therefore, by the residue theorem and (3.30), we have

$$I_2(y,s)=\frac{\pi}{2\pi i}\int_{(c')}\Gamma(s-1+z)\zeta(z)\cot\left(\frac{\pi z}{2}\right)\left(\frac{4\pi^2k}{y}\right)^{-z}dz. \tag{3.30}$$

Replace z by $s-1+z$ and x by $4\pi^2xk/y$ in (3.28), then use the resulting equation, (3.1) and Parseval's formula to obtain, for $1-\mathrm{Re}(s)<c''<0$,

$$\frac{\pi}{2\pi i}\int_{(c'')}\Gamma(s-1+z)\zeta(z)\cot\left(\frac{\pi z}{2}\right)\left(\frac{4\pi^2k}{y}\right)^{-z}dz=\pi\ \mathrm{PV}\int_0^\infty\left(\frac{4\pi^2xk}{y}\right)^{s-1}e^{-\frac{4\pi^2xk}{y}}\cot(\pi x)dx. \quad (3.31)$$

The existence of the principal value integral appearing on the right-hand side of (3.31) is shown by Hardy [25, p. 31]. Hence, from (3.30) and (3.31),

$$I_2(y,s)=\pi\ \mathrm{PV}\int_0^\infty\left(\frac{4\pi^2xk}{y}\right)^{s-1}e^{-\frac{4\pi^2xk}{y}}\cot(\pi x)dx. \quad (3.32)$$

Substituting (3.27) and (3.32) in (3.22) as well as in (3.25), we get

$$\int_0^\infty\cos\left(\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt=\left(\frac{2\pi k}{y}\right)^{1-s}\Bigg\{\cos\left(\frac{\pi s}{2}\right)\left(\pi\left(\frac{4\pi^2k}{y}\right)^{s-1}\sum_{n=1}^\infty n^{s-1}e^{-\frac{4\pi^2nk}{y}}-\frac{y\Gamma(s)}{4\pi k}\right)$$
$$+\sin\left(\frac{\pi s}{2}\right)\pi\ \mathrm{PV}\int_0^\infty\left(\frac{4\pi^2xk}{y}\right)^{s-1}e^{-\frac{4\pi^2xk}{y}}\cot(\pi x)dx\Bigg\}, \quad (3.33)$$

and

$$\int_0^\infty\sin\left(\frac{2\pi kt}{y}\right)\frac{t^{s-1}}{e^t-1}dt=\left(\frac{2\pi k}{y}\right)^{1-s}\Bigg\{\sin\left(\frac{\pi s}{2}\right)\left(\pi\left(\frac{4\pi^2k}{y}\right)^{s-1}\sum_{n=1}^\infty n^{s-1}e^{-\frac{4\pi^2nk}{y}}-\frac{y\Gamma(s)}{4\pi k}\right)$$
$$-\cos\left(\frac{\pi s}{2}\right)\pi\ \mathrm{PV}\int_0^\infty\left(\frac{4\pi^2xk}{y}\right)^{s-1}e^{-\frac{4\pi^2xk}{y}}\cot(\pi x)dx\Bigg\}. \quad (3.34)$$

Substituting (3.33) and (3.34) in (3.20) and simplifying, we are led to

$$\sum_{n=1}^\infty\frac{(n-a)^{s-1}}{e^{(n-a)y}-1}=\frac{\Gamma(s)\zeta(s)}{y^s}+\left(\frac{2\pi}{y}\right)^s\cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^\infty n^{s-1}\sum_{k=1}^\infty\cos(2\pi ak)\,e^{-\frac{4\pi^2nk}{y}}-\frac{\Gamma(s)}{(2\pi)^s}\cos\left(\frac{\pi s}{2}\right)\sum_{k=1}^\infty\frac{\cos(2\pi ak)}{k^s}$$
$$-\left(\frac{2\pi}{y}\right)^s\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^\infty n^{s-1}\sum_{k=1}^\infty\sin(2\pi ak)\,e^{-\frac{4\pi^2nk}{y}}+\frac{\Gamma(s)}{(2\pi)^s}\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^\infty\frac{\sin(2\pi ak)}{k^s}$$
$$+\left(\frac{2\pi}{y}\right)^s\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^\infty\cos(2\pi ak)\mathrm{PV}\int_0^\infty x^{s-1}e^{-\frac{4\pi^2xk}{y}}\cot(\pi x)dx$$
$$+\left(\frac{2\pi}{y}\right)^s\cos\left(\frac{\pi s}{2}\right)\sum_{k=1}^\infty\sin(2\pi ak)\mathrm{PV}\int_0^\infty x^{s-1}e^{-\frac{4\pi^2xk}{y}}\cot(\pi x)dx. \quad (3.35)$$

Invoking Lemma 3.3 we can interchange the summation and integration for the last expression on the right-hand side of (3.35). Also note that one can justify the same for the series involving $\cos(2\pi ak)$. Hence, after simplification, (3.35) becomes

$$\sum_{n=1}^\infty\frac{(n-a)^{s-1}}{e^{(n-a)y}-1}=\frac{\Gamma(s)\zeta(s)}{y^s}+\left(\frac{2\pi}{y}\right)^s\sum_{n=1}^\infty n^{s-1}\sum_{k=1}^\infty\cos\left(\frac{\pi s}{2}+2\pi ak\right)e^{-4\pi^2nk/y}-\frac{\Gamma(s)}{(2\pi)^s}\sum_{k=1}^\infty\frac{\cos\left(\frac{\pi s}{2}+2\pi ak\right)}{k^s}$$
$$+\left(\frac{2\pi}{y}\right)^s\mathrm{PV}\int_0^\infty\left(\sum_{k=1}^\infty\sin\left(\frac{\pi s}{2}+2\pi ak\right)e^{-4\pi^2kx/y}\right)x^{s-1}\cot(\pi x)dx.$$

Making the change of variable $x \to xy/(2\pi)$ in the integral and rearranging the terms in the above expression, we get

$$\frac{\Gamma(s)\zeta(s)}{y^s}+\left(\frac{2\pi}{y}\right)^s\sum_{n=1}^{\infty}n^{s-1}\sum_{k=1}^{\infty}\cos\left(\frac{\pi s}{2}+2\pi ak\right)e^{-4\pi^2nk/y}$$
$$=\frac{\Gamma(s)}{(2\pi)^s}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi s}{2}+2\pi ak\right)}{k^s}+\sum_{n=1}^{\infty}\frac{(n-a)^{s-1}}{e^{(n-a)y}-1}-\mathrm{PV}\int_0^{\infty}\left(\sum_{k=1}^{\infty}\sin\left(\frac{\pi s}{2}+2\pi ak\right)e^{-2\pi kx}\right)x^{s-1}\cot\left(\frac{1}{2}yx\right)dx. \tag{3.36}$$

Now, using the fact $\cos(\theta)=(e^{i\theta}+e^{-i\theta})/2$, we find

$$\sum_{k=1}^{\infty}\cos\left(\frac{\pi s}{2}+2\pi ak\right)e^{-4\pi^2nk/y}=\frac{1}{2}e^{\pi is/2}\sum_{k=1}^{\infty}e^{-\left(\frac{4\pi^2n}{y}-2\pi ia\right)k}+\frac{1}{2}e^{-\pi is/2}\sum_{k=1}^{\infty}e^{-\left(\frac{4\pi^2n}{y}+2\pi ia\right)k}$$
$$=\frac{1}{2}\left(\frac{e^{\pi is/2}}{e^{\frac{4\pi^2n}{y}-2\pi ia}-1}+\frac{e^{-\pi is/2}}{e^{\frac{4\pi^2n}{y}+2\pi ia}-1}\right). \tag{3.37}$$

Similarly,

$$\sum_{k=1}^{\infty}\sin\left(\frac{\pi s}{2}+2\pi ak\right)e^{-2\pi kx}=\frac{1}{2i}\left(\frac{e^{\pi is/2}}{e^{2\pi x-2\pi ia}-1}-\frac{e^{-\pi is/2}}{e^{2\pi x+2\pi ia}-1}\right). \tag{3.38}$$

Substituting (3.37) and (3.38) in (3.36), we get

$$\frac{\Gamma(s)\zeta(s)}{y^s}+\left(\frac{2\pi}{y}\right)^s\frac{1}{2}\sum_{n=1}^{\infty}n^{s-1}\left(\frac{e^{\pi is/2}}{e^{\frac{4\pi^2n}{y}-2\pi ia}-1}+\frac{e^{-\pi is/2}}{e^{\frac{4\pi^2n}{y}+2\pi ia}-1}\right)$$
$$=\frac{\Gamma(s)}{(2\pi)^s}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi s}{2}+2\pi ak\right)}{k^s}+\sum_{n=1}^{\infty}\frac{(n-a)^{s-1}}{e^{(n-a)y}-1}$$
$$-\frac{1}{2i}\mathrm{PV}\int_0^{\infty}x^{s-1}\left(\frac{e^{\pi is/2}}{e^{2\pi x-2\pi ia}-1}-\frac{e^{-\pi is/2}}{e^{2\pi x+2\pi ia}-1}\right)\cot\left(\frac{1}{2}yx\right)dx.$$

Finally, we arrive at (1.3) after multiplying by $(2\pi/y)^{-s}$ on the both sides of the above equation and then letting $4\pi^2/y=\alpha$ with $\alpha\beta=4\pi^2$. □

Proof of Corollary 1.2. Let $a=1/2$ and $s=2m, m\in\mathbb{N}$ in Theorem 1.1 and observe that the principal value integral vanishes. The result then follows upon using Euler's formula [43, p. 5, Equation (1.14)]

$$\zeta(2m)=(-1)^{m+1}\frac{(2\pi)^{2m}B_{2m}}{2(2m)!}. \tag{3.39}$$

□

Proof of Corollary 1.3. For an even integer m, we have [12, p. 23] (also see [2, p. 25, Exercise 15(c)])

$$\sum_{n=1}^{\infty}\frac{(2n-1)^{2m+1}}{e^{\pi(2n-1)}+1}=(2^{2m+1}-1)\frac{B_{2m+2}}{4m+4}.$$

The first result follows upon letting $\alpha=\beta=2\pi$ and m to be a positive odd integer in Corollary 1.4 and then using the above evaluation after replacing m by $m-1$. The second one follows by using the elementary identity

$$\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$$

and Glaisher's evaluation [20]

$$\sum_{n=1}^{\infty}\frac{n^{2m-1}}{e^{2n\pi}-1}=\frac{B_{2m}}{4m}.$$

□

Proof of Corollary 1.4. Letting $s=2m$ and $a=1/4$ in Theorem 1.1 and simplifying, we get

$$\alpha^m\left\{\frac{\Gamma(2m)\zeta(2m)}{(2\pi)^{2m}}+(-1)^{m+1}\sum_{n=1}^{\infty}\frac{n^{2m-1}}{e^{2n\alpha}+1}\right\}$$
$$=\beta^m\left\{\frac{(-1)^m\Gamma(2m)}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{\cos(\pi k/2)}{k^{2m}}+\sum_{n=1}^{\infty}\frac{(n-1/4)^{2m-1}}{e^{(n-1/4)\beta}-1}-\frac{(-1)^m}{2}\int_0^{\infty}\frac{x^{2m-1}\cot(\beta x/2)}{\cosh(x)}dx\right\}. \quad (3.40)$$

Now take $s=2m$ and $a=3/4$ in Theorem 1.1 to obtain

$$\alpha^m\left\{\frac{\Gamma(2m)\zeta(2m)}{(2\pi)^{2m}}+(-1)^{m+1}\sum_{n=1}^{\infty}\frac{n^{2m-1}}{e^{2n\alpha}+1}\right\}$$
$$=\beta^m\left\{\frac{(-1)^m\Gamma(2m)}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{(-1)^k\cos(\pi k/2)}{k^{2m}}+\sum_{n=1}^{\infty}\frac{(n-3/4)^{2m-1}}{e^{(n-3/4)\beta}-1}+\frac{(-1)^m}{2}\int_0^{\infty}\frac{x^{2m-1}\cot(\beta x/2)}{\cosh(x)}dx\right\}. \quad (3.41)$$

Now add (3.40) and (3.41) so that

$$\alpha^m\left\{2\frac{\Gamma(2m)\zeta(2m)}{(2\pi)^{2m}}+2(-1)^{m+1}\sum_{n=1}^{\infty}\frac{n^{2m-1}}{e^{2n\alpha}+1}\right\}$$
$$=\beta^m\left\{\frac{(-1)^m\Gamma(2m)}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{(1+(-1)^k)\cos(\pi k/2)}{k^{2m}}+\sum_{n=1}^{\infty}\frac{(n-1/4)^{2m-1}}{e^{(n-1/4)\beta}-1}+\sum_{n=1}^{\infty}\frac{(n-3/4)^{2m-1}}{e^{(n-3/4)\beta}-1}\right\}.$$

Using the fact $\sum_{n=1}^{\infty}(-1)^k/k^{2m}=2^{-2m}(2-2^{2m})\zeta(2m)$ in the above equation, we arrive at (1.5).

Next, subtracting (3.41) from (3.40) yields

$$\frac{(-1)^m\Gamma(2m)}{(2\pi)^{2m}}\sum_{k=1}^{\infty}\frac{(1-(-1)^k)\cos(\pi k/2)}{k^{2m}}+\sum_{n=1}^{\infty}\frac{(n-1/4)^{2m-1}}{e^{(n-1/4)\beta}-1}-\sum_{n=1}^{\infty}\frac{(n-3/4)^{2m-1}}{e^{(n-3/4)\beta}-1}$$
$$=(-1)^m\int_0^{\infty}\frac{x^{2m-1}\cot(\beta x/2)}{\cosh(x)}dx.$$

Note that the first sum on the left-hand side of the above equation vanishes. Therefore, rewriting the left-hand side in terms of the Dirichlet character χ defined in (1.7), we are led to (1.6). □

4. A simple proof of the transformation for $\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny}$

In [16], this theorem was obtained for the first time as a corollary of a more general result, namely, (1.9). Hence the absolute convergence of the series on the right-hand side of (1.11) resulted automatically. In what follows, we not only give a direct proof of this result, but also prove from scratch the convergence of the series.

To that end, we first prove the identity for $y > 0$ and later extend it to $\mathrm{Re}(y) > 0$ by analytic continuation. We begin by showing the absolute convergence of the series on the right-hand side of (1.11). Note that for $w > 0$, (1.20) and Theorem 1.12 imply

$$\sinh(w)\mathrm{Shi}(w) - \cosh(w)\mathrm{Chi}(w) + \sum_{j=1}^{m}(2j-1)!w^{-2j} = \mathfrak{R}_m(1, w) = \frac{1}{w^{2m}}\mathfrak{R}_m(w, 1). \tag{4.1}$$

Now employ Lemma 2.1 for $\mathfrak{R}_m(w, 1)$, and then let $w = 4\pi^2 n/y$, where $y > 0$ (as assumed at the beginning of the proof), so that as $n \to \infty$, we have

$$\sinh\left(\frac{4\pi^2 n}{y}\right)\mathrm{Shi}\left(\frac{4\pi^2 n}{y}\right) - \cosh\left(\frac{4\pi^2 n}{y}\right)\mathrm{Chi}\left(\frac{4\pi^2 n}{y}\right) + \sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2 n}{y}\right)^{-2j} = O_{m,y}\left(\frac{1}{n^{2m+2}}\right). \tag{4.2}$$

The absolute convergence of $\sum_{n=1}^{\infty}\sigma_{2m}(n)/n^{2m+2}$ then implies that of the series on the right-hand side of (1.11) with the help of the above estimate.

We now prove (1.11). Let $a = 0$ and $s = 2m+1$ in (3.18) so that

$$\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny} = \frac{(2m)!}{y^{2m+1}}\left\{\zeta(2m+1) + \sum_{k=1}^{\infty}\left(\zeta\left(1+2m, 1+\frac{2\pi ik}{y}\right) + \zeta\left(1+2m, 1-\frac{2\pi ik}{y}\right)\right)\right\}.$$

Using (2.22), we can see that for $m \in \mathbb{N}$,

$$\zeta\left(1+2m, 1+\frac{2\pi ik}{y}\right) + \zeta\left(1+2m, 1-\frac{2\pi ik}{y}\right) = \zeta\left(1+2m, \frac{2\pi ik}{y}\right) + \zeta\left(1+2m, -\frac{2\pi ik}{y}\right).$$

Now employ Theorem 1.9 with $z = m$ and $w = 2\pi k/y$ in the above equation to see that

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny} &= \frac{(2m)!}{y^{2m+1}}\Bigg\{\zeta(2m+1) + \frac{\cos(\pi m)}{m}\left(\frac{2\pi}{y}\right)^{-2m}\sum_{k=1}^{\infty}\frac{1}{k^{2m}} \\ &\quad + 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\infty}\left(\frac{1}{(v-2\pi ik/y)^{2m+1}} + \frac{1}{(v+2\pi ik/y)^{2m+1}}\right)\cos(2\pi nv)\,dv\Bigg\} \\ &= \frac{(2m)!}{y^{2m+1}}\Bigg\{\zeta(2m+1) + \frac{(-1)^m}{m}\left(\frac{2\pi}{y}\right)^{-2m}\zeta(2m) + 2(2\pi)^{2m}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}n^{2m} \\ &\quad \times\int_0^{\infty}\left(\frac{1}{(t-4\pi^2 ink/y)^{2m+1}} + \frac{1}{(t+4\pi^2 ink/y)^{2m+1}}\right)\cos(t)\,dt\Bigg\},\end{aligned}$$

where in the last step we made change of variable $v = t/(2\pi n)$. Letting $nk = \ell$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny} &= \frac{(2m)!}{y^{2m+1}}\Bigg\{\zeta(2m+1) + \frac{(-1)^m}{m}\left(\frac{2\pi}{y}\right)^{-2m}\zeta(2m) + 2(2\pi)^{2m}\sum_{\ell=1}^{\infty}\sum_{n|\ell}n^{2m} \\ &\quad \times\int_0^{\infty}\left(\frac{1}{(t-4\pi^2 i\ell/y)^{2m+1}} + \frac{1}{(t+4\pi^2 i\ell/y)^{2m+1}}\right)\cos(t)\,dt\Bigg\}.\end{aligned} \tag{4.3}$$

Next, invoke Theorem 1.12 with $w = 4\pi^2\ell/y$ in (4.3) to arrive at

$$\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny} = \frac{(2m)!}{y^{2m+1}}\Bigg\{\zeta(2m+1) + \frac{(-1)^m}{m}\left(\frac{2\pi}{y}\right)^{-2m}\zeta(2m) + \frac{4(-1)^m(2\pi)^{2m}}{(2m)!}\sum_{\ell=1}^{\infty}\sigma_{2m}(\ell)$$

$$\times\left\{\sinh\left(\frac{4\pi^2\ell}{y}\right)\mathrm{Shi}\left(\frac{4\pi^2\ell}{y}\right)-\cosh\left(\frac{4\pi^2\ell}{y}\right)\mathrm{Chi}\left(\frac{4\pi^2\ell}{y}\right)+\sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2\ell}{y}\right)^{-2j}\right\}. \tag{4.4}$$

Using (3.39) in (4.4) and rearranging the terms leads to (1.11) for $y>0$. The result can be extended by analytic continuation to $\mathrm{Re}(y)>0$. This is seen as follows. Clearly, the left-hand side of (1.11) is analytic in this region. We now show that the series on the right is also analytic. In order to prove this using Weierstrass' theorem on analytic functions, we need only show that (4.2) holds for $\mathrm{Re}(y)>0$ as well. To that end, employing $(1-\sqrt{\xi})^{-(2m+1)}-(1+\sqrt{\xi})^{-(2m+1)}=2(2m+1)\sqrt{\xi}\,{}_2F_1\left(m+1,m+\frac{3}{2};\frac{3}{2};\xi\right)$, we find that

$$\int_0^{\infty}\left(\frac{1}{(t-iw)^{2m+1}}+\frac{1}{(t+iw)^{2m+1}}\right)\cos(t)dt=\frac{2(2m+1)}{(-1)^m w^{2m+2}}\int_0^{\infty}t\cos(t){}_2F_1\left(m+1,m+\frac{3}{2};\frac{3}{2};-\frac{t^2}{w^2}\right)dt. \tag{4.5}$$

The integral on the right can be evaluated in terms of the Meijer G-function $G_{2,4}^{3,1}\left(\begin{matrix}1,\frac{3}{2}\\1,m+1,m+\frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{w^2}{4}\right)$ employing [19, p. 81, Formula **8.17.6**]. That

$$G_{2,4}^{3,1}\left(\begin{matrix}1,\frac{3}{2}\\1,m+1,m+\frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{w^2}{4}\right)=-\frac{w^2}{\sqrt{\pi}2^{2m+1}}\sum_{j=1}^{r+1}\frac{\Gamma(2m+2j)}{w^{2j}}+O(w^{-2r-4}),\quad \text{as } w\to\infty,\ \mathrm{Re}(w)>0 \tag{4.6}$$

can then be obtained using the asymptotic of this Meijer G-function given in [31, p. 179, Theorem 2]. With $w=4\pi^2n/y, \mathrm{Re}(y)>0$, the first equality of (4.1) and (4.5) finally prove (4.2). This completes the proof of (1.11) for $\mathrm{Re}(y)>0$.

□

5. Asymptotics of the plane partitions generating function

Proof of Corollary 1.6. The following estimate can be directly shown to hold as $y\to 0$ in $\mathrm{Re}(y)>0$ as is done later. However, we first prove it separately for real $y\to 0^+$ owing to its simplicity.

Indeed, for real $y\to 0^+$, Lemma 2.1 along with (4.1) imply that

$$\sinh\left(\frac{4\pi^2n}{y}\right)\mathrm{Shi}\left(\frac{4\pi^2n}{y}\right)-\cosh\left(\frac{4\pi^2n}{y}\right)\mathrm{Chi}\left(\frac{4\pi^2n}{y}\right)+\sum_{j=1}^{m}(2j-1)!\left(\frac{4\pi^2n}{y}\right)^{-2j}$$
$$=\frac{(-1)^{m+1}}{(4\pi^2n)^{2m}}y^{2m}\sum_{j=1}^{r+1}\frac{\Gamma(2m+2j)}{(4\pi^2n)^{2j}}y^{2j}+O\left(y^{2r+2m+4}\right). \tag{5.1}$$

For complex y in $\mathrm{Re}(y)>0$ such that $y\to 0$, (5.1) is seen to hold from the first equality of (4.1), (4.5) and (4.6).

Substituting (5.1) in Theorem 1.5, we deduce that

$$\sum_{n=1}^{\infty}\sigma_{2m}(n)e^{-ny}=\frac{(2m)!}{y^{2m+1}}\zeta(2m+1)-\frac{B_{2m}}{2my}-\frac{2}{\pi}\left(\frac{2\pi}{y}\right)^{2m+1}\frac{y^{2m}}{(2\pi)^{4m}}\sum_{n=1}^{\infty}\frac{\sigma_{2m}(n)}{n^{2m}}\sum_{j=1}^{r+1}\frac{\Gamma(2m+2j)}{(4\pi^2n)^{2j}}y^{2j}$$
$$+O\left(y^{2r+3}\right)$$

$$= \frac{(2m)!}{y^{2m+1}}\zeta(2m+1) - \frac{B_{2m}}{2my} - \frac{2}{y\pi(2\pi)^{2m-1}}\sum_{j=1}^{r+1}\frac{\Gamma(2m+2j)}{(4\pi^2)^{2j}}y^{2j}\sum_{n=1}^{\infty}\frac{\sigma_{2m}(n)}{n^{2m+2j}} + O\left(y^{2r+3}\right).$$

Using the well-known identity $\sum_{n=1}^{\infty}\sigma_a(n)n^{-s} = \zeta(s)\zeta(s-a)$, where $\mathrm{Re}(s) > 1, \mathrm{Re}(s-a) > 1$, in the above expression, we arrive at (1.13). □

We are now ready to derive Wright's result from [46] as a special case of Corollary 1.6.

Proof of Corollary 1.15. Letting $m = 1$ and $y = \log(1/x), |x| < 1$ in Corollary 1.7 and using (1.14), as $x \to 1^-$, we have

$$x\frac{d}{dx}\log F(x) = -\frac{2\zeta(3)}{(\log x)^3} + \frac{1}{12\log x} + \frac{1}{\pi^2}\sum_{j=1}^{r+1}\frac{\Gamma(2j+2)\zeta(2j+2)\zeta(2j)}{(2\pi)^{4j}}(\log x)^{2j-1} + O\left(-(\log x)^{2r+3}\right). \tag{5.2}$$

Now divide both sides by x and then integrate with respect to x to get

$$\log F(x) = c + \frac{\zeta(3)}{(\log x)^2} + \frac{1}{12}\log\log x + \frac{1}{\pi^2}\sum_{j=1}^{r+1}\frac{\Gamma(2j+2)\zeta(2j+2)\zeta(2j)}{2j(2\pi)^{4j}}(\log x)^{2j} + O\left((\log x)^{2r+4}\right), \tag{5.3}$$

where c is an integrating constant. Exponentiating both sides of the above equation, we arrive at (1.15). □

6. Concluding remarks

For general a with $0 \le a < 1$, the generalized Lambert series

$$\sum_{n=1}^{\infty}\frac{(n-a)^{s-1}}{e^{nz}-1} \qquad (s \in \mathbb{C}, \mathrm{Re}(z) > 0)$$

does not seem to have been studied before. It makes its appearance for the first time in Theorem 1.1 of our paper. It may be interesting to undertake a further study of this series.

In [11, Theorem 1], Bradley obtains a generalization of Ramanujan's formula (1.1) for periodic functions g with period $m \in \mathbb{N}$. When g is even, his transformation involves the series of the type $\sum_{n=1}^{\infty}\frac{g(n)n^{-2m-1}}{e^{n\beta}-1}$ whereas for g odd, it involves $\sum_{n=1}^{\infty}\frac{g(n)n^{-2m}}{e^{n\beta}-1}$. Observe that the series in our (1.6) involves $\sum_{n=1}^{\infty}\frac{\chi(n)n^{2m-1}}{e^{n\beta}-1}$, where $\chi(n)$ defined in (1.7) is an odd Dirichlet character and $m \in \mathbb{N}, m > 1$, and hence does not fall under the purview of Bradley's transformation. Thus it may be worthwhile to see if a more general transformation encompassing our series exists. We note that another series which is not covered by Bradley's transformation is $\sum_{n=1}^{\infty}\frac{n^{-2m}}{e^{n\beta}-1}$, for which a transformation was recently obtained in [16, Theorem 2.12].

In [16], Theorem 1.5 was obtained as a special case of (1.9) by tour de force whereas in the present paper, this has been accomplished directly. One can then ask if a direct proof of Theorem 2.12 of [16], which is a transformation for $\sum_{n=1}^{\infty}\frac{n^{-2m}}{e^{n\beta}-1}$, can be derived without resorting to (1.9).

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