$$= \frac{1}{\sqrt{2}} e^{\left(-\frac{\pi}{4} \mp 2n\pi\right)} \cdot \left(\cos\left(-\frac{\pi}{4} \mp 2n\pi + \frac{1}{2}\ln(2)\right) + i \sin\left(-\frac{\pi}{4} \mp 2n\pi + \frac{1}{2}\ln(2)\right) \right).$$

Ch. 14.1 – Line integral in the complex plane

- Complex definite integrals are called (complex) line integrals, denoted by $\int_C f(z)\,dz$, meaning $f(z)$ (the integrand) is integrated over a given curve C in $\mathbb{C}$ known as the path of integration,
- $C: \quad z(t) = x(t) + i y(t) \quad (a \le t \le b)$

  (Parametric representation)
- C is a smooth curve if its derivative $\dot{z} = \frac{dz}{dt}$ is continuous & non-zero at each point

Definition of the Complex line integral

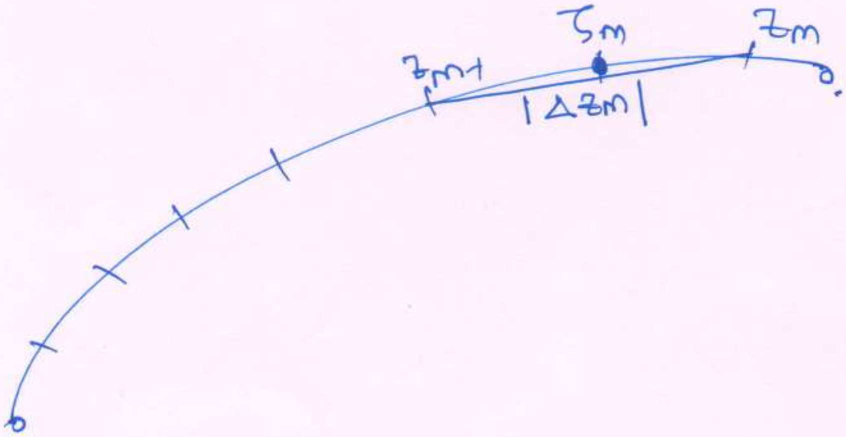
Let C be a smooth curve in the complex plane given by $z(t) = x(t) + i y(t)$, $a \le t \le b$, and let f be continuous at each point of C. Partition the interval $a \le t \le b$ by points $t_0 (= a), t_1, \dots, t_n (= b)$, where $t_0 < t_1 < \dots < t_n$.

Let the corresponding sub-division of C be $z_0, z_1, \ldots, z_n \, (= Z)$, where $z_j = z(t_j)$.

Choose ζ_1 on part of C between z_0 & z_1 (i.e. $\zeta_1 = z(t)$, $t_0 \le t \le t_1$), and similarly, ζ_2 between z_1 and z_2, etc.

Let $S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m$, where $\Delta z_m = z_m - z_{m-1}$.

We do a similar thing independently with $n = 2, 3, \ldots$, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$.



- $|\Delta z_m|$ also approaches zero:
  - Arc length of the smooth curve C is a continuous function of t.
  - Since $|t_m - t_{m-1}| \to 0$, $z_{m-1} \to z_m$, hence $|\Delta z_m| \to 0$.

- Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{m=1}^{n} f(\zeta_m)\Delta z_m$ is called the line integral of $f(z)$ over the oriented curve C.

This is

Assumption: All paths of integration for complex line integrals are assumed to be piecewise smooth, that is, consist of finitely many smooth curves joined end to end.

Linearity

- $$\int_C (c_1 f_1(z) + c_2 f_2(z))\, dz = c_1 \int_C f_1(z)\, dz + c_2 \int_C f_2(z)\, dz$$

Sense reversal

- $$\int_{z_0}^{Z} f(z)\, dz = -\int_{Z}^{z_0} f(z)\, dz$$

Partitioning the path

- $$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

Existence of the complex line integral

The line integral $\int_C f(z)\, dz$ exists if f is continuous and C is piecewise smooth:

Let $f(z) = u(x,y) + iv(x,y)$.

$$\zeta_m = \xi_m + i\eta_m \quad \& \quad \Delta z_m = \Delta x_m + i\Delta y_m$$

$$\Rightarrow S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m$$

$$= \sum_{m=1}^{n} (u+iv)(\Delta x_m + i\Delta y_m),$$

where $u = u(\xi_m, \eta_m)$, $v = v(\xi_m, \eta_m)$

$$\Rightarrow S_n = \sum u\Delta x_m - \sum v\Delta y_m + i\left(\sum u\Delta y_m + \sum v\Delta x_m\right)$$

which are real sums,

- Now f continuous $\Rightarrow$ u and v are continuous
- $n \to \infty$ implies the greatest Δx_m & $\Delta y_m \to 0$

Hence we have each sum as real line integral:

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz = \int_C u\,dx - \int_C v\,dy + i\left(\int_C u\,dy + \int_C v\,dx\right).$$

Since f is continuous & C is piecewise smooth, each real line integral on the right exists.

$\Rightarrow \int_C f(z)\,dz$ exists.

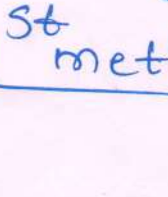
- Moreover, its value is independent of the choice of subdivisions & intermediate points ζ_m.

Evaluating definite integrals

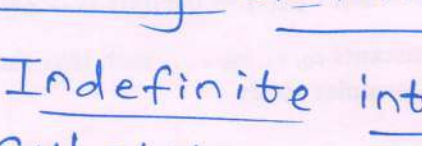
1st method: Indefinite integration and Substitution of limits

- Analogue of $\int_a^b f(x)\,dx = F(b) - F(a)$ $\quad \left(F'(x) = f(x)\right)$
- Suitable for analytic functions only.

Defn. A domain D is called simply connected if every simple closed curve (closed curve without self-intersections) encloses only points of D.

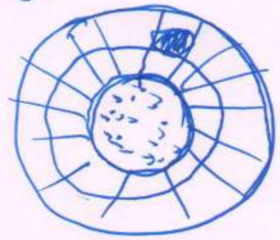


closed curve without self-intersection



closed curve with self-intersection

- Circular disk is simply connected but an annulus is not.



- A simply connected domain is one where one can continuously shrink any simple closed curve into a point, while remaining in the domain.

Indefinite integration of analytic functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic fn. $F(z)$ s.t. $F'(z) = f(z)$ in D, & for all paths in D joining two points z_0 & z_1 in D, we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0), \quad [F'(z) = f(z)]$$

- Remark: D for an entire function f is the whole complex plane.

Eg ① $\displaystyle\int_0^i z^3\,dz = \left[\frac{z^4}{4}\right]_0^i = \frac{1}{4} - 0 = \frac{1}{4}.$

② $\displaystyle\int_{-\pi i}^{\pi i} \sinh z\,dz = \left[\cosh z\right]_{-\pi i}^{\pi i} = \cosh(\pi i) - \cosh(-\pi i) = 0.$

③ $\displaystyle\int_{-3\pi i}^{e^{\frac{3\pi i}{4}}} \frac{1}{z}\,dz = \mathrm{Ln}\left(e^{\frac{3\pi i}{4}}\right) - \mathrm{Ln}\left(e^{-\frac{3\pi i}{4}}\right) = \frac{3\pi i}{4} - \left(-3\frac{\pi i}{4}\right) = \frac{3\pi i}{2}$