Use of a representation of a path

- Applies not only to analytic functions but also to any continuous complex function.

Thm. Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt}, \quad \text{where } \dot{z} = \frac{dz}{dt}$$

Proof: Note that, for $z = x+iy$, & $f(z) = u(x(t), y(t)) + iv(x(t), y(t))$,

$$\int_C f(z)\,dz = \int_C u\,dx - \int_C v\,dy + i\left(\int_C u\,dy + \int_C v\,dx\right) \quad \text{— (1)}$$

~~Let~~ Note that $\dot{z} = \dot{x} + i\dot{y}$.

Also $dx = \dot{x}\,dt$ & $dy = \dot{y}\,dt$.

Thus, $\displaystyle\int_a^b f(z(t))\,\dot{z}(t)\,dt = \int_a^b (u+iv)(\dot{x}+i\dot{y})\,dt$

$$= \int_a^b (u+iv)(dx+i\,dy)$$

$$= \int_a^b \big((u\,dx - v\,dy) + i\,(u\,dy + v\,dx)\big)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx). \quad \text{— (2)}$$

This proves the result (compare (1) & (2)).

Steps in applying the above result

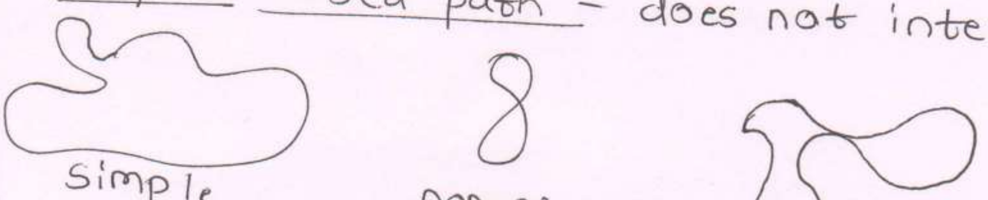
1. Represent the path C in the form $z(t)$ $(a \le t \le b)$.
2. Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$.
3. Substitute $z(t)$ for every z in $f(z)$.
4. Integrate $f(z(t))\dot{z}(t)$ over t from a to b.

Eg. Prove that $\oint_C \frac{1}{z}\, dz = 2\pi i$, where C is the unit circle traversed in the counter-clockwise direction.

Cauchy's integral theorem

- In general, a line integral of a function $f(z)$ depends not only on the endpoints of the path, but also on the choice of the path itself.
- However, if $f(z)$ is analytic in a simply connected domain D, then the integral is path-independent.

- Simple closed path – does not intersect or touch itself.



simple   non-simple   non-simple

- Simply connected domain – defn. already seen

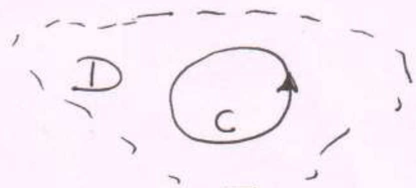
doubly connected

triply connected

Thm. (Cauchy's integral theorem)

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0 .$$



[simple closed path = contour]

Examples ① $\oint_C e^z\,dz=0$, $\oint_C \sin z\,dz=0$, $\oint_C z^n\,dz=0$ $n\in\mathbb{N}\cup\{0\}$.

② Singularities outside contour

- $\oint_C \sec z\,dz=0$ C: unit circle

- $\oint_C \frac{dz}{z^2+9}=0$ C: unit circle.

③ Non-analytic fn.

$\oint_C \bar{z}\,dz = \int e^{-it}\cdot e^{it}\cdot i\,dt = 2\pi i \neq 0$.
(unit circle)

④ Analyticity sufficient, not necessary)

$\oint_C \frac{dz}{z^2}=0$, where C is unit circle.

⑤ Simple connectedness essential

$\oint_C \frac{1}{z}\,dz = 2\pi i$. C: unit circle. (counter-clockwise)

CAUCHY'S PROOF OF THE ABOVE THEOREM
(with additional assumption on continuity of $f'(z)$)

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx).$$

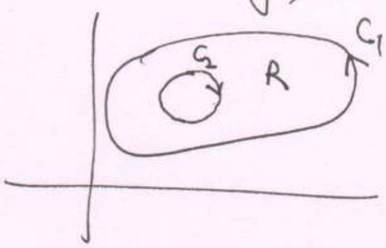
$f(z)$ analytic in D $\Rightarrow$ $f'(z)$ exists in D.
$f'(z)$ continuous $\Rightarrow$ u & v have continuous partial derivatives.

This follows from the fact that

$$f'(z) = u_x + iv_x \quad \& \quad f'(z) = -iu_y + v_y.$$

Then we apply Green's theorem which states that if R is a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves, and if $F_1(x,y)$ and $F_2(x,y)$ are functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ & $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R, then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1 dx + F_2 dy).$$



Let $F_1 = u$ & $F_2 = -v$. Then

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

(region bounded by C)

But $v_x = -u_y \Rightarrow \oint_C (u\,dx - v\,dy) = 0$.

Similarly $u_x = v_y \Rightarrow \oint_C (u\,dy + v\,dx) = 0$.

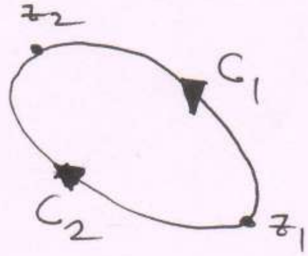
$$\Rightarrow \oint_C f(z)\,dz = 0.$$

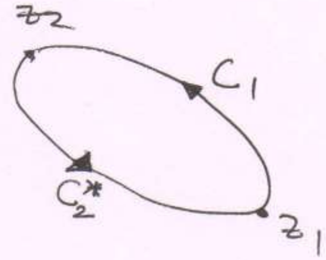
<u>Remark</u>: Goursat proved Cauchy's theorem without assuming that $f'(z)$ is continuous.

PATH INDEPENDENCE

Thm. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Proof:



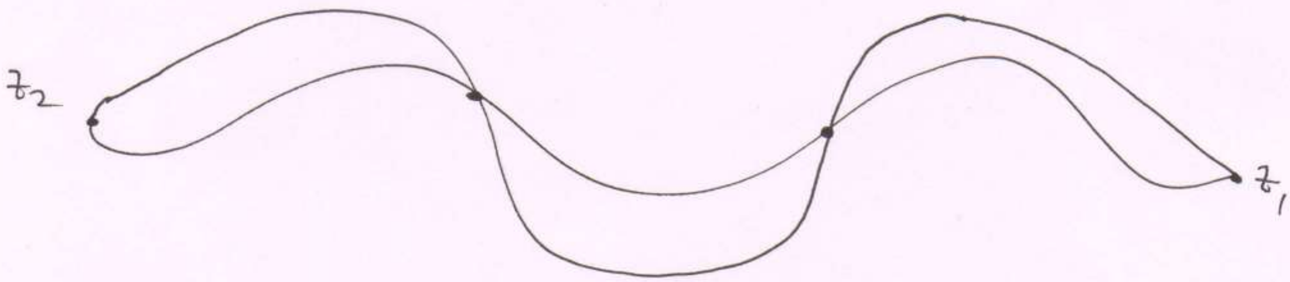


$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz.$$

$$= \int_{C_2} f(z)\,dz. \quad \blacksquare$$

Generalization



- Principle of deformation of path

Let D be simply connected & f analytic on D. The line integral of $f(z)$ over a path between points z_0 & z_1 in D retains the same value when we continuously deform it within D keeping z_0 & z_1 fixed.

- Cauchy's theorem for multiply connected domains

A is <u>sufficient</u> for B if $A \Rightarrow B$

A is <u>necessary</u> for B if $B \Rightarrow A$

$$\oint_C \frac{1}{z^2}\, dz \qquad z = e^{it}, \quad 0 \le t \le 2\pi$$

$$dz = i e^{it}\, dt$$

$$\frac{1}{z^2} = e^{-2it}$$

$$= \int_0^{2\pi} e^{-2it} \cdot i e^{it}\, dt = \int_0^{2\pi} i e^{-it}\, dt$$

$$= i \int_0^{2\pi} (\cos t - i \sin t)\, dt$$

$$= i \left(\underline{\int_0^{2\pi} \cos t\, dt} - i \underline{\int_0^{2\pi} \sin t\, dt} \right)$$

$$= i(0 - i0)$$

$$= 0.$$