Section 14.4 – Derivatives of Analytic functions

Thm. 1 If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are analytic functions in D too. The values of these derivatives at a point z_0 in D are given by

$$\boxed{\begin{aligned} f'(z_0) &= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^2}, \\ f''(z_0) &= \frac{2!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^3}, \end{aligned}}$$

and, in general,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n \in \mathbb{N})},$$

where C is any simple closed curve (path) in D enclosing z_0 and whose complete interior belongs to D. We integrate in a ~~clock~~ counter-clockwise direction.

Examples ① $\displaystyle\oint_{C \text{ (unit circle)}} \frac{e^{-z}\sin z}{z^2}\,dz = \oint_C \frac{e^{-z}\sin z}{(z-0)^2}\,dz$

$$= 2\pi i f'(0), \text{ where}$$

$f(z) = e^{-z}\sin z$.

Hence $f'(z) = e^{-z}(\cos z - \sin z)$ so that

$f'(0) = 1$

$$\Rightarrow \oint_C \frac{e^{-z}\sin z}{z^2}\,dz = 2\pi i.$$

2) $$\oint_C \frac{z^6}{(2z-1)^6}\,dz = \frac{1}{2^6}\oint_C \frac{z^6}{(z-\frac{1}{2})^6}\,dz$$
(unit circle)

$$= \frac{1}{2^6}\cdot\frac{2\pi i}{5!}\, f^{(5)}(z)\Big|_{z=\frac{1}{2}},\quad \text{where } f(z)=z^6.$$

$$f^{(5)}(z) = 6!\,z \Rightarrow f^{(5)}(z)\Big|_{z=1/2} = \frac{6!}{2} \text{ so that}$$

$$\oint_C \frac{z^6}{(2z-1)^6}\,dz = \frac{3\pi i}{32}.$$

3) $$\oint_C \frac{\operatorname{Ln}(z+3)}{(z-2)(z+1)^2}\,dz \quad (C:\text{ boundary of the square with vertices } \pm 1.5, \pm 1.5i)$$

$$= 2\pi i\, f'(-1), \text{ where}$$

$$f(z) = \frac{\operatorname{Ln}(z+3)}{z-2},$$

$$\boxed{\text{Ans. } 2\pi i\left(-\frac{1}{6} - \frac{\operatorname{Ln}(2)}{9}\right)}$$

Corollary

Cauchy's inequality

If $f(z)$ is analytic in domain D, and C is a ~~simple closed path~~ circle of radius r enclosing $z_0 \in D$ such that $|f(z)| \le M$ for all z on C, and $M > 0$, then

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!\,M}{r^n}.}$$

Proof: By Thm. 1,

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z)|}{r^{n+1}} . |i r e^{i\theta} d\theta| \qquad (z = z_0 + re^{i\theta})$$

$$\leq \frac{n!}{2\pi} \cdot \frac{M}{r^n} \cdot \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{n! \, M}{r^n}.$$

∎

Thm. 2 $\boxed{\text{Liouville's theorem}}$

Any bounded entire function must be a constant.

Proof: Let $|f(z)| \leq K$ $\forall z$. From the above corollary, $|f'(z_0)| \leq \frac{K}{r}$. Since $f(z)$ is entire, this is true for every r, so we can take r as large as we want. Thus, letting $r \to \infty$, we see that

$0 \leq |f'(z_0)| \leq 0 \implies f'(z_0) = 0$

$\implies f'(z) = 0 \quad \forall z$ ($\because z_0$ was an arbitrary point)

$\implies f(z)$ is constant.

∎

Morera's theorem (Converse of Cauchy's integral thm.)

Thm. 3 If $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z)\,dz = 0$ for every closed path in D, then $f(z)$ is analytic in D.

Proof of Theorem 1

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

(Since f is analytic on D, it is differe--ntiable at all points of D.)

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz\right].$$

(by Cauchy's integral theorem.)

Now

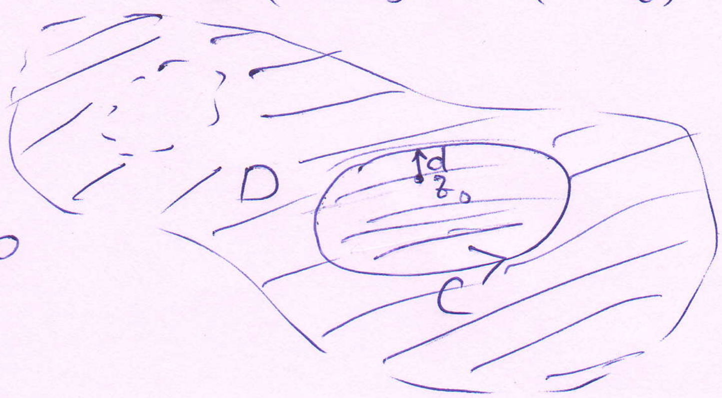
$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0-\Delta z)(z-z_0)} \qquad \text{(A)}$$

We now have to show that as $\Delta z \to 0$, the above integral tends to $\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^2}$.

To that end, consider the difference of the two integrals:

$$\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz - \oint_C \frac{f(z)}{(z-z_0)^2}\,dz = \oint_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz$$

We will now use ML-ineq. to show that the integral on the right approaches zero as $\Delta z \to 0$.



- Since f is analytic on C, it is continuous on C; hence bounded in absolute value, say $|f(z)| \leq K$.
- Let d be the smallest distance from z_0 to the points of C. Then $\forall z \in C$,
- $|z-z_0|^2 \geq d^2 \Rightarrow \frac{1}{|z-z_0|^2} \leq \frac{1}{d^2}$.

By Δ - inequality, $\forall z \in C$,

$$d \leq |z - z_0| = |z - z_0 - \Delta z + \Delta z| \leq |z - z_0 - \Delta z| + |\Delta z|.$$

Let $\frac{1}{2} d \geq \Delta z$ so that

$$\frac{1}{2} d \leq d - |\Delta z| \leq |z - z_0 - \Delta z|$$

$$\Rightarrow \frac{1}{|z - z_0 - \Delta z|} \leq \frac{2}{d}.$$

Let L be the length of C. If $|\Delta z| \leq d/2$, by the ML - inequality,

$$\left| \oint_C \frac{f(z)\, \Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz \right| \leq K L |\Delta z| \cdot \frac{2}{d} \cdot \frac{1}{d^2}$$

$$\longrightarrow 0 \text{ as } \Delta z \to 0.$$

Thus $\oint_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)(z - z_0)} \xrightarrow{\Delta z \to 0} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$ —— B

From A & B, the result now follows.