⑥ Now $\frac{e^{iz}-1}{z}$ has a removable singularity at $z=0$

$\Rightarrow \exists\, M$ (constant), $M>0 \ni \left|\frac{e^{iz}-1}{z}\right| \le M$ for $|z|\le 1$,

$$\Rightarrow \left|\int_{\gamma_r} \frac{e^{iz}-1}{z}\,dz\right| \le \pi r M \quad \text{so that}$$

$$\lim_{r\to 0}\int_{\gamma_r} \frac{e^{iz}-1}{z}\,dz = 0 \quad \text{⑤}$$

which gives from ④, $\lim_{r\to 0}\int_{\gamma_r} \frac{e^{iz}}{z}\,dz = -\pi i$

Thus from ①, ②, ③ and ⑤,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{R\to\infty}\lim_{r\to 0}\int_r^R \frac{\sin x}{x}\,dx$$

$$= \frac{1}{2i}\left\{-\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{iz}}{z}\,dz - \lim_{r\to 0}\frac{e^{iz}}{z}\,dz\right\}$$

$$= \frac{1}{2i}\left(-0-(-\pi i)\right) = \frac{\pi}{2}.$$

③ For $a>1$, show that $\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$.

__Proof:__ Let $z=e^{i\theta}$, then $\bar{z}=e^{-i\theta}=1/z$.

and $a+\cos\theta = a+\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = a+\frac{1}{2}\left(z+\frac{1}{z}\right)$

$$= \frac{z^2+2az+1}{2z},$$

$$\Rightarrow \int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$
(again using, $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x) = f(2a-x)$)

$$= -i\int_\gamma \frac{dz}{z^2+az+1} \quad (\because dz = ie^{i\theta}d\theta = iz\,d\theta)$$

$$= -i\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)}, \quad \text{--- (1)}$$

where $\alpha = -a+\sqrt{a^2-1}$, $\beta = -a-\sqrt{a^2-1}$.

Now $a>1 \Rightarrow 2a>2 \Rightarrow a^2-2a+1 < a^2-1$
$\Rightarrow a-1 < \sqrt{a^2-1}$ so that $-1 < -a+\sqrt{a^2-1}$

Similarly, one can show $-a+\sqrt{a^2-1} < 1$

$\Rightarrow |\alpha| < 1$

Similarly $|\beta|>1$. Hence we see that only one of the 2 roots of z^2+az+1 lie inside γ.

By residue theorem,

$$\int_\gamma \frac{dz}{z^2+az+1} = 2\pi i \operatorname{Res}\left(\frac{1}{(z-\alpha)(z-\beta)}; \alpha\right)$$

$$= 2\pi i \lim_{z\to\alpha} \frac{z-\alpha}{(z-\alpha)(z-\beta)}$$

$$= \frac{2\pi i}{\alpha-\beta} = \frac{2\pi i}{2\sqrt{a^2-1}} \quad \text{--- (2)}$$

From (1) & (2),

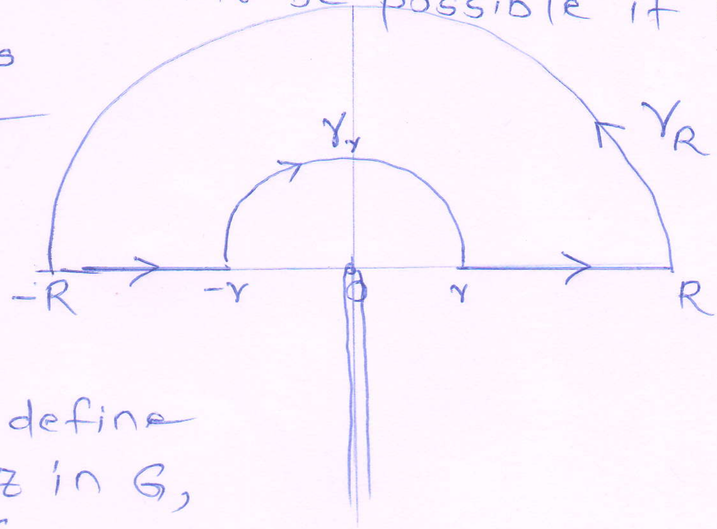
$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$$

[1]

④ Prove that $\int_0^\infty \frac{\log x \, dx}{1+x^2} = 0$.

Proof: Note that if $f(z) = \frac{\log z}{1+z^2}$, we would want to capture the possible contributions of the residues at the poles of f at $z = \pm i$.

But this wouldn't be possible if we choose the contour as follows & use the principal branch of $\log z$.



Hence we define $\log z$ for z in G, where $G = \{z \in \mathbb{C} : z \neq 0, \ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\}$

Let $l(z) = \log|z| + i\theta$ for $z = |z|e^{i\theta} \neq 0$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$,

Let $0 < r < R$ & refer to the above contour.

Note that $l(x) = \log x$ for $x > 0$ & $l(x) = \log|x| + \pi i$ for $x < 0$.

Hence

$$\int_\gamma \frac{l(z)}{1+z^2} dz = \int_r^R \frac{\log x}{1+x^2} dx + iR \int_0^\pi \frac{(\log R + i\theta)}{1+R^2 e^{2i\theta}} e^{i\theta} d\theta$$

$$+ \int_{-R}^{-r} \frac{(\log|x| + \pi i)}{1+x^2} dx + ir \int_\pi^0 \frac{(\log r + i\theta)}{1+r^2 e^{2i\theta}} e^{i\theta} d\theta. \quad \text{—— (1)}$$

Now the only pole of $\frac{l(z)}{1+z^2}$ lying inside γ is at $z = i$, which is actually a simple pole with residue

$$\lim_{z \to i} \frac{(z-i)\, l(z)}{(z-i)(z+i)} = \frac{l(i)}{2i} = \frac{\log|i| + \frac{\pi}{2} i}{2i} = \frac{\pi}{4}.$$

$$\Rightarrow \int_\gamma \frac{l(z)}{1+z^2}\,dz = \frac{\pi^2 i}{2}$$

Also, $\displaystyle\int_r^R \frac{\log x}{1+x^2}\,dx + \underbrace{\int_{-R}^{-r} \frac{\log|x| + \pi i}{1+x^2}\,dx}_{\text{In the 1st part involving just } \log|x|, \text{ replace } x \text{ by } -x}$

$$= 2\int_r^R \frac{\log x}{1+x^2}\,dx + \pi i \int_r^R \frac{dx}{1+x^2}.$$

Now letting $r \to 0^+$ & $R \to \infty$ and using the fact

$$\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1}(x)\right]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

it follows from (1),

$$\frac{\pi^2 i}{2} = 2\lim_{\substack{r\to 0^+\\ R\to\infty}} \int_r^R \frac{\log x}{1+x^2}\,dx + \pi i\left(\frac{\pi}{2}\right) + \lim_{R\to\infty} iR\int_0^\pi \frac{(\log R + i\theta)e^{i\theta}\,d\theta}{1+R^2e^{2i\theta}}$$

$$+ \lim_{r\to 0^+} ir\int_\pi^0 \frac{(\log r + i\theta)}{1+r^2e^{2i\theta}}\,e^{i\theta}\,d\theta.$$

$$\Rightarrow \int_0^\infty \frac{\log x}{1+x^2}\,dx = \frac{1}{2}\lim_{r\to 0^+} ir\int_0^\pi \frac{(\log r + i\theta)e^{i\theta}\,d\theta}{1+r^2e^{2i\theta}}$$

$$- \frac{1}{2}\lim_{R\to\infty} iR\int_0^\pi \frac{(\log R + i\theta)e^{i\theta}\,d\theta}{1+R^2e^{2i\theta}}.$$

Now if $\rho > 0$, & since $|1+\rho^2e^{i\theta}| \geq |1-\rho^2|$,

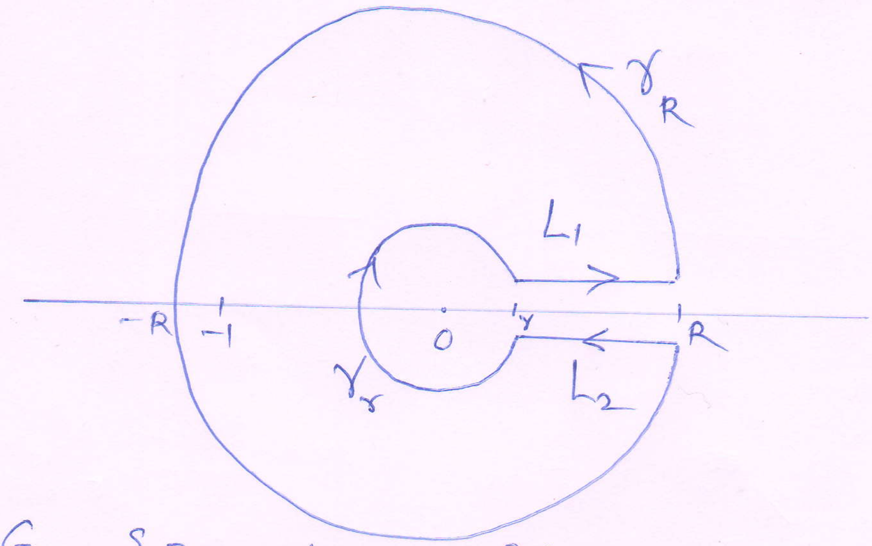
$$\left|\rho\int_0^\pi \frac{(\log\rho + i\theta)e^{i\theta}\,d\theta}{1+\rho^2e^{i\theta}}\right| \leq \frac{\rho|\log\rho|}{|1-\rho^2|}\int_0^\pi d\theta + \frac{\rho}{|1-\rho^2|}\int_0^\pi \theta\,d\theta$$

$$= \frac{\pi\rho|\log\rho|}{|1-\rho^2|} + \frac{\pi^2\rho}{2|1-\rho^2|} \longrightarrow 0 \text{ as } \rho \to 0^+ \text{ or } \rho \to \infty.$$

$$\Rightarrow \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0,$$

5) For $0<c<1$, prove that $\int_0^\infty \frac{x^{-c}}{1+x}\,dx = \frac{\pi}{\sin \pi c}$.

Proof: The branch of the function z^{-c} should be accounted for while evaluating this integral, Note that $z=0$ is called a branch point of z^{-c} (i.e., a point around which, if we take an arbitrarily small nbhd, the function is discontinuous while going over that nbhd.)



Let $G = \{z : z \neq 0 \text{ and } 0 < \arg z < 2\pi\}$. We define a branch of the logarithm on G by putting $l(re^{i\theta}) = \log r + i\theta$ where $0<\theta<2\pi$. For $z \in G$, we put

$$f(z) = \exp(-c\,l(z)) \quad (f \text{ is a branch of } z^{-c})$$

Let $\gamma = \gamma_R + L_2 + \gamma_r + L_1$ be a closed curve enclosing -1, where $0<r<1<R$, $\delta>0$, with $L_1 = [r+\delta i, R+\delta i]$ and $L_2 = [R-\delta i, r-\delta i]$ (considering the orientation)

Note that $\gamma \approx 0$ in G and

$$\operatorname{Res}\left(\frac{f(z)}{1+z}; -1\right) = \lim_{z\to -1} \frac{(z+1)z^{-c}}{1+z} = (-1)^{-c} = e^{-\pi i c}$$

($\because \arg(-1) = \pi$, as $0<\arg z<2\pi$)

$$\Rightarrow \int_\gamma \frac{f(z)}{1+z}\,dz = 2\pi i\, e^{-i\pi c} \quad \text{--- (1)}$$

Next, by the defn. of the line integral,

$$\int_{L_1} \frac{f(z)}{1+z}\,dz = \int_{r}^{R} \frac{f(t+i\delta)}{1+t+i\delta}\,dt$$

We now show that $\lim_{\delta\to 0^+} \int_{L_1} \frac{f(z)}{1+z}\,dz = \int_{r}^{R} \frac{t^{-c}}{1+t}\,dt$.

Define $g(t,\delta)$ on the compact set $[r,R]\times[0,\pi/2]$ by

$$g(t,\delta) = \begin{cases} \left|\frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t}\right|, & \delta>0 \\ 0, & \delta = 0 \end{cases}.$$

Then g is continuous on $[r,R]\times[0,\pi/2]$, hence uniformly continuous. So given $\varepsilon>0$, $\exists\ \delta_0>0 \ni$ if $\sqrt{(t-t')^2+(\delta-\delta')^2} < \delta_0$, then

$$|g(t,\delta)-g(t',\delta')| < \varepsilon/R.$$

In particular, for $t=t'$ and $\delta'=0$, we have

$g(t,\delta) < \varepsilon/R$ whenever $r\le t\le R$ and $\delta<\delta_0$.

$\Rightarrow \int_{r}^{R} g(t,\delta)\,dt \le \varepsilon$. for $\delta<\delta_0$.

This implies $\int_{r}^{R} \frac{t^{-c}}{1+t}\,dt = \lim_{\delta\to 0^+} \int_{L_1} \frac{f(z)}{1+z}\,dz$ ——— (2)

Now $l(\bar z) = \overline{l(z)} + 2\pi i$. Hence one can similarly show

$$-e^{-2\pi i c}\int_{r}^{R} \frac{t^{-c}}{1+t}\,dt = \lim_{\delta\to 0^+} \int_{L_2} \frac{f(z)}{1+z}\,dz . \quad \text{——— (3)}$$

Note that the integral in (1) is independent of δ. Hence letting $\delta\to 0^+$ in (1) & using (2) & (3), we see that

$$2\pi i e^{-i\pi c} = \lim_{\delta\to 0^+} \int_{\gamma} \frac{f(z)}{1+z}\, dz$$

$$= \lim_{\delta\to 0^+} \int_{\gamma_R} \frac{f(z)}{1+z}\, dz + \lim_{\delta\to 0^+} \int_{L_2} \frac{f(z)}{1+z}\, dz + \lim_{\delta\to 0} \int_{\gamma_r} \frac{f(z)}{1+z}\, dz$$

$$+ \lim_{\delta\to 0^+} \int_{L_1} \frac{f(z)}{1+z}\, dz .$$

$$\left.\begin{aligned} \Rightarrow 2\pi i e^{-i\pi c} &- (1-e^{-2\pi i c}) \int_r^R \frac{t^{-c}}{1+t}\, dt \\ &= \lim_{\delta\to 0^+} \left[\int_{\gamma_r} \frac{f(z)}{1+z}\, dz + \int_{\gamma_R} \frac{f(z)}{1+z}\, dz \right] \end{aligned}\right\} \quad (4)$$

Now if $\rho > 0$ & $\rho \neq 1$, and if γ_ρ is the part of the circle $|z| = \rho$ from $\sqrt{\rho^2-\delta^2} + i\delta$ to $\sqrt{\rho^2-\delta^2} - i\delta$ then

$$\left| \int_{\gamma_\rho} \frac{f(z)}{1+z}\, dz \right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho$$

Since the RHS of the above ineq. is independent of δ, from (4),

$$\left| 2\pi i e^{-i\pi c} - (1-e^{-2\pi i c}) \int_r^R \frac{t^{-c}}{1+t}\, dt \right| \leq \frac{r^{-c}}{|1-r|} 2\pi r + \frac{R^{-c}}{|1-R|} 2\pi R .$$

Now let $R \to \infty$ and $r \to 0^+$. RHS $\to 0$. Hence

$$2\pi i e^{-i\pi c} = (1-e^{-2\pi i c}) \int_0^\infty \frac{t^{-c}}{1+t}\, dt$$

$$\Rightarrow \int_0^\infty \frac{t^{-c}}{1+t}\, dt = \frac{2\pi i e^{-i\pi c}}{e^{-\pi i c}(e^{\pi i c} - e^{-\pi i c})} = \frac{\pi}{\left(\frac{e^{\pi i c} - e^{-\pi i c}}{2i}\right)}$$

$$= \frac{\pi}{\sin \pi c} .$$

$\boxtimes$