Sect. 15.2 - Power series

- Very important in complex analysis
- Every analytic function can be represented by power series.
- Their sums are analytic functions.

Defn. A power series in powers of $z-z_0$ is a series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots,}$$

where $\underbrace{z}_{\text{(variable)}} \in \mathbb{C}$, $\underbrace{a_i}_{\text{(constants)}} \in \mathbb{C}$, for $0 \le i < \infty$, called coefficients of the series, and $\underbrace{z_0}_{\text{(constant)}} \in \mathbb{C}$ called the center of the series.

Special case:

$$\boxed{\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots}$$

* Power series may converge in a disk with center z_0 or in the whole z-plane or only at z_0.

Examples

① The geometric series $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$ converges absolutely for $|z|<1$ & diverges for $|z| \geq 1$.

② $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$ converges absolutely for every z since by root test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\,\frac{n!}{z^n}\right|$$

$$= \lim_{n\to\infty}\frac{|z|}{n+1}$$

$$= 0 < 1.$$

(3) $\sum_{n=0}^{\infty} n!\,z^n = 1+z+2z^2+6z^3+\cdots$

converges only at $z=0$. since

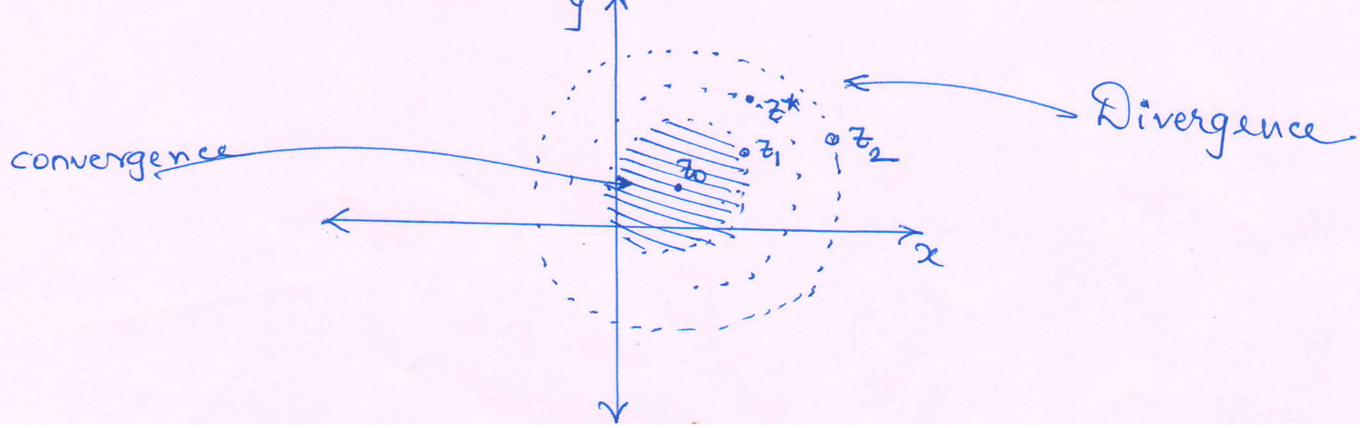
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,z^{n+1}}{n!\,z^n}\right| = |z|\lim_{n\to\infty}(n+1) = \infty > 1,$$

Thm. 1 (Convergence of a power series)

(a) Every power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at the center z_0.

(b) If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at a point $z=z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, that is, $|z-z_0| < |z_1-z_0|$.

(c) If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges at a $z=z_2$, it diverges for every z farther away from z_0 than z_2.



Proof: (a) At $z=z_0$, $\sum_{n=0}^{\infty} a_n(z_0-z_0)^n = a_0 + \sum_{n=1}^{\infty} a_n(z_0-z_0)^n = a_0$, hence convergent.

(b) Convergence at $z=z_1 \Rightarrow a_n(z_1-z_0)^n \to 0$ as $n\to\infty$.

$\Rightarrow \exists\, M>0 \ni |a_n(z_1-z_0)^n| < M \ \forall n\in\mathbb{N}$.

Now
$$|a_n(z-z_0)^n| = \left| a_n(z_1-z_0)^n \left(\frac{z-z_0}{z_1-z_0}\right)^n \right|$$
$$< M \left|\frac{z-z_0}{z_1-z_0}\right|^n . \qquad (*)$$

Now by the hypothesis, $|z-z_0|<|z_1-z_0|$. Hence $\sum_{n=0}^{\infty}\left|\frac{z-z_0}{z_1-z_0}\right|^n$ converges (geometric series).

$\Longrightarrow \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely by (*) & comparison test.

∎

(c) If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges at $z=z_2$, but converges for some z farther away from z_0 than z_0, then (b) implies $\sum_{n=0}^{\infty} a_n(z_2-z_0)^n$ converges, which is a contradiction.

∎

Radius of Convergence of a power series

— is the radius of the ~~smallest~~ largest circle with center z_0 such that the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all points within this circle.

- $|z-z_0|=R$ is called the circle of convergence.
- Since R is as large as possible, the series diverges $\forall z$ with $|z-z_0|>R$

- On the circle of convergence itself, nothing can be said: the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ may converge at some or all or none of the points.

e.g. The series $\sum \frac{z^n}{n^2}$, $\sum \frac{z^n}{n}$ & $\sum z^n$ all have radius of convergence $R=1$. However, for z on the circle $|z|=1$,

- $\left|\sum \frac{z^n}{n^2}\right| \leq \sum \frac{|z|^n}{n^2} = \sum \frac{1}{n^2}$ converges $\forall z$.
- $\sum \frac{z^n}{n}$ converges for $z=-1$ (Leibnitz test), but diverges for $z=1$ (harmonic series). (In fact, it converges for all $z \ni |z|=1$ & $z \neq 1$.
- $\sum z^n = \sum e^{in\theta}$ diverges everywhere on $|z|=1$.

Thm. 2 (Finding the radius of convergence R — the Cauchy-Hadamard formula)

Suppose $\left|\frac{a_{n+1}}{a_n}\right|$, $n \in \mathbb{N}$ converges with limit L^*.

- If $L^*=0$, $R=\infty$, that is, the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges $\forall z$.
- If $L^* \neq 0$ (and hence $L^*>0$), we have

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

Proof :- By ratio test, the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z s.t.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n (z-z_0)^n} \right| < 1.$$

However, $L = \left(\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right) \cdot |z-z_0| = L^* |z-z_0|.$

Hence, The series converges for all z with $|z-z_0| < \frac{1}{L^*}$ & diverges $\forall\, z$ with $|z-z_0| > \frac{1}{L^*}$.

So that $R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$.

Examples Find the center and the radius of convergence of the following power series.

① $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$ $\quad R = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \lim_{n\to\infty} \frac{1}{(1+1/n)} = 1.$

② $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$ $\quad R = \lim_{n\to\infty} \left| \frac{(3n)!}{2^n (n!)^3} \cdot \frac{2^{n+1}((n+1)!)^3}{(3n+3)!} \right|$

$$= \lim_{n\to\infty} \left| \frac{2(n+1)^3}{(3n+3)(3n+2)(3n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2\left(1+1/n\right)^3}{\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)} \right|$$

$$= \frac{2}{}$$

③ Example of a series which is NOT a power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

$$b_n = \frac{(-1)^n}{n} z^{n(n+1)}$$

$$L = \lim_{n\to\infty} |b_n|^{1/n} < 1 \Rightarrow \text{series converges}$$

$$\lim_{n\to\infty} |b_n|^{1/n} = \lim_{n\to\infty} \frac{|z|^{n+1}}{n^{1/n}} = 0 < 1, \text{ provided } |z|<1$$

So this series converges for $|z|<1$.
& diverges for $|z|>1$.