Important Taylor series / Maclaurin series

① If $f(z) = \frac{1}{1-z}$.

$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ so that $f^{(n)}(0) = n!$

$$\Rightarrow \boxed{\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot n! = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots,}$$

when $|z|<1$.

② Exponential function

$f(z) = e^z \Rightarrow f^{(n)}(z) = e^z \;\forall n \in \mathbb{N}$. Thus,

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots}$$

Note that if $z = iy$, then

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)$$

$= \cos y + i \sin y$.

Thus we get Euler's formula $e^{iy} = \cos y + i \sin y$.

③ Trigonometric & Hyperbolic functions

i) $$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left\{\left(1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \dots\right) + \left(1 - iz + \frac{(-iz)^2}{2!} + \frac{(-iz)^3}{3!} + \dots\right)\right\}$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

ii) $$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

(iii) $\cosh z = \cos iz = \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

(iv) $\sinh z = -i \sin(iz) = -i \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

④ <u>Logarithm</u>

$$\operatorname{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad (|z|<1)$$

<u>Proof:</u> $\operatorname{Ln}(1+z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \frac{f^{(n)}(0)}{n!}$

and $f(z) = \operatorname{Ln}(1+z)$.

$f(0) = \operatorname{Ln}(1+0) = 0 \quad \Rightarrow a_0 = 0$

$f'(0) = \left.\frac{1}{1+z}\right|_0 = 1 \quad \Rightarrow a_1 = \frac{1}{1!} = 1$

$f''(0) = \left.\frac{-1}{(1+z)^2}\right|_0 = -1 \Rightarrow a_2 = \frac{-1}{2!} = \frac{-1}{2}$.

$\vdots$

In general,

$f^{(n)}(0) = \left.\frac{(-1)(-2)(-3)\cdots(-(n-1))}{(1+z)^n}\right|_0 = \left.\frac{(-1)^{n-1}(n-1)!}{(1+z)^n}\right|_0$

$= (-1)^{n-1}(n-1)!$

$\Rightarrow a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n}. \Rightarrow \boxed{\operatorname{Ln}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}}$

Hence $\boxed{\operatorname{Ln}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}}$

Replacing z by $-z$, we see that

$$\operatorname{Ln}(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

$$= -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4}\right) + \cdots$$

$$\Rightarrow \operatorname{Ln}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \qquad (|z| < 1$$

(5) Combining the series for $\operatorname{Ln}(1+z)$ & $\operatorname{Ln}(1-z)$, we see that

$$\operatorname{Ln}\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1)$$

In Class test 1, we saw that

$$\tanh^{-1}(z) = \frac{1}{2} \operatorname{Ln}\left(\frac{1+z}{1-z}\right).$$

Hence

$$\tanh^{-1}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} \qquad (|z| < 1)$$

Different methods for finding Taylor & Maclaurin series

(1)
$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n \qquad |z| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

(2) Integrating both sides w.r.t. z, we see that

$$\int \frac{1}{1+z^2}\, dz = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1} + c$$

$$\Rightarrow \tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1} + c$$

Let $z = 0$ so that $c = 0$.

$$\Rightarrow \tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots \qquad (|z| < 1)$$

This series represents the principal value $w = u + iv = \tan^{-1} z$, that is the value for which $|u| < \pi/2$.

(3) Develop $\frac{1}{c-z}$ in powers of $z - z_0$, where $c - z_0 \neq 0$.

$$\frac{1}{c-z} = \frac{1}{(c-z_0)-(z-z_0)} = \frac{1}{(c-z_0)\left(1-\frac{z-z_0}{c-z_0}\right)} = \frac{1}{c-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{c-z_0}\right)^n,$$

if $\left|\frac{z-z_0}{c-z_0}\right| < 1$. Thus, in this case,

$$\frac{1}{c-z} = \frac{1}{(c-z_0)}\left\{1 + \frac{z-z_0}{c-z_0} + \left(\frac{z-z_0}{c-z_0}\right)^2 + \cdots\right\}.$$

(4) Find the Taylor series of the function

$f(z) = \frac{5z^2+2z-1}{z^3-5z^2+z-5}$ with center $z_0 = 0$.

Ans. We first expand $f(z)$ in partial fractions:

$$f(z) = \frac{Az+B}{z^2+1} + \frac{C}{z-5} = \frac{-\frac{2}{13}z + \frac{16}{13}}{1+z^2} + \frac{\frac{67}{13}}{z-5}$$

$$=: (1) + (2), \text{ say.}$$

Now

$$(1) = \frac{1}{13}\frac{(-2z+1)}{1+z^2} = \frac{-2z+1}{13}\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$$= \frac{-2}{13}\sum_{n=0}^{\infty}(-1)^n z^{2n+1} + \frac{1}{13}\sum_{n=0}^{\infty}(-1)^n z^{2n} \quad \text{for } |z|<1$$

Also

$$(2) = \frac{67/13}{z-5} = \frac{67/13}{-5(1-z/5)} = \frac{-5\times 67}{13}\sum_{n=0}^{\infty}(z/5)^n \quad \text{for } |z|<5.$$

$$\Rightarrow \frac{5z^2+2z-1}{z^3-5z^2+z-5} = \frac{-2}{13}\sum_{n=0}^{\infty}(-1)^n z^{2n+1} + \frac{1}{13}\sum_{n=0}^{\infty}(-1)^n z^{2n} - \frac{5\times 67}{13}\sum_{n=0}^{\infty}\frac{z^n}{5^n}.$$

for $|z|<1$.

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Binomial Theorem

$$(1+z)^n = \sum_{k=0}^{n}\binom{n}{k} z^k \text{, when } n \in \mathbb{N}\cup\{0\}.$$

What about n negative ?

Generalized-binomial theorem:

$$\frac{1}{(1+z)^m} = \sum_{n=0}^{\infty}\binom{-m}{n} z^n = 1 - mz + \frac{m(m+1)}{2!}z^2 - \frac{m(m+1)(m+2)}{3!}z^3 + \dots$$

for $|z|<1$.

Ch. 16.1 – Laurent series & residue integration

- **Laurent series** — contain both non-negative integer powers of $z-z_0$ (like the Taylor series) and negative integer powers of $z-z_0$.

Thm. 1 Laurent's theorem

If $f(z)$ is analytic on two concentric circles C_1 & C_2 with center z_0 and in the annulus between them, then