Zeros of analytic functions

A zero of an analytic function f at $z = z_0$ has order n if not only f is zero at that point, but also $f', f'', \ldots f^{(n-1)}$ are zero at $z = z_0$, and $f^{(n)}(z_0) \neq 0$.

- A first order zero is called a simple zero.

Eg. (i) $(z+a)^4$ has a fourth-order zero at $z = -a$.

(ii) $1+z^2$ has simple zeros at $\pm i$.

(iii) e^z has no zeros.

(iv) $1-\cos z$ has second-order zeros at $0, \pm 2\pi, \pm 4\pi, \ldots$

Taylor series at a zero:

If f has an m^{th}-order zero at $z = z_0$, then its Taylor series is given by

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n, \quad (a_m \neq 0)$$

This is characteristic of such a zero, because if $f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$, by differentiation, it follows that f has an m^{th}-order zero.

Thm. (Zeros)

The zeros of an analytic function $f(z)\ (\not\equiv 0)$ are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

Proof: If f has an m^{th} order zero at $z = z_0$, then

$$f(z) = (z-z_0)^m (a_m + a_{m+1}(z-z_0) + \cdots), \text{ where } a_m \neq 0.$$

Note that $(z-z_0)^m$ is zero only at $z=z_0$.
The power series $a_m + a_{m+1}(z-z_0)+\cdots$ represents an analytic function, say $g(z)$. Note that $g(z_0)=a_m \neq 0$.
Since an analytic function is continuous, and hence $g(z)\neq 0$ in some neighborhood of $z=z_0$ also, so is true with $f(z)$.

∎

Thm. (Poles & zeros)

Let $f(z)$ be analytic at $z=z_0$ and have a zero of n^{th} order at $z=z_0$. Then $1/f(z)$ has a pole of n^{th} order at $z=z_0$.

The same holds for $\frac{h(z)}{f(z)}$ if $h(z)$ is analytic at $z=z_0$ & $h(z_0)\neq 0$.

Analyticity / Singularity at infinity

If we want to investigate $f(z)$ for large $|z|$, set $z=1/w$ and investigate $f(z)=f(1/w)=g(w)$ in a neighborhood of $w=0$.

$f(z)$ is defined as analytic or singular at infinity if $g(w)$ is analytic or singular, ~~at~~ resp; at $w=0$.

If $\lim\limits_{w\to 0} g(w)$ exists, we define $g(0)=\lim\limits_{w\to 0} g(w)$.

- We say $f(z)$ has an n^{th} order zero at infinity if $f(1/w)$ has such a zero at $w=0$. Similarly for poles and essential singularities

Eg. $f(z)=1/z^2$ analytic at $\infty \Leftrightarrow f(\frac{1}{z}) = z^2$ analytic at 0.

- $f(z)=z^3$ singular at ∞.

$\Updownarrow$

$f(\frac{1}{z})=\frac{1}{z^3}$ singular at 0.

Stereographic projection & compactification