Zeros of analytic functions

A zero of an analytic function f at $z=z_0$ has order n if not only f is zero at that point, but also $f', f'', \ldots f^{(n-1)}$ are zero at $z=z_0$, and $f^{(n)}(z_0) \neq 0$.

- A first order zero is called a simple zero.

Eg. (i) $(z+a)^4$ has a fourth-order zero at $z=-a$.
(ii) $1+z^2$ has simple zeros at $\pm i$.
(iii) e^z has no zeros.
(iv) $1-\cos z$ has second-order zeros at $0, \pm 2\pi, \pm 4\pi, \ldots$

Taylor series at a zero:

If f has an m^{th}-order zero at $z=z_0$, then its Taylor series is given by

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n, \quad (a_m \neq 0)$$

This is characteristic of such a zero, because if $f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$, by differentiation, it follows that f has an m^{th}-order zero.

Thm. (Zeros)

The zeros of an analytic function $f(z)$ $(\not\equiv 0)$ are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

Proof: If f has an m^{th} order zero at $z=z_0$, then

$$f(z) = (z-z_0)^m (a_m + a_{m+1}(z-z_0) + \cdots), \text{ where } a_m \neq 0.$$

Note that $(z-z_0)^m$ is zero only at $z=z_0$.
The power series $a_m + a_{m+1}(z-z_0) + \cdots$ represents an analytic function, say $g(z)$. Note that $g(z_0) = a_m \neq 0$.
Since an analytic function is continuous, and hence $g(z) \neq 0$ in some neighborhood of $z=z_0$ also, so is true with $f(z)$.

Thm. (Poles & zeros)

Let $f(z)$ be analytic at $z=z_0$ and have a zero of n^{th} order at $z=z_0$. Then $1/f(z)$ has a pole of n^{th} order at $z=z_0$.

The same holds for $\frac{h(z)}{f(z)}$ if $h(z)$ is analytic at $z=z_0$ & $h(z_0) \neq 0$.

Analyticity / Singularity at infinity

If we want to investigate $f(z)$ for large $|z|$, set $z = 1/w$ and investigate $f(z) = f(1/w) = g(w)$ in a neighborhood of $w=0$.

$f(z)$ is defined as analytic or singular at infinity if $g(w)$ is analytic or singular, ~~at~~ resp; at $w=0$.

If $\lim\limits_{w\to 0} g(w)$ exists, we define $g(0) = \lim\limits_{w\to 0} g(w)$.

- We say $f(z)$ has an n^{th} order zero at infinity if $f(1/w)$ has such a zero at $w=0$. Similarly for poles and essential singularities

Eg. $f(z) = 1/z^2$ analytic at $\infty \Rightarrow f(\frac{1}{z}) = z^2$
$f(z) = z^3$ singular at ∞.
$\Downarrow$
$f(\frac{1}{z}) = \frac{1}{z^3}$

- A non-constant entire fn., being unbounded, has a pole if it's a polynomial or an essential singularity if it is not.

- An analytic fn. whose only singularities in the finite plane are poles is called meromorphic fn, e.g. cosec(z),

Sect. 16.3 – Residue Integration Method

- If f is analytic on and inside a simple closed path C, then $\oint_C f(z)\,dz = 0$ by Cauchy's integral theorem.
- If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C & inside it, $f(z)$ admits a Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

which converges in $0 < |z - z_0| < R$ for some positive R.

MAIN IDEA

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz .$$

Now there are many ways to obtain Laurent series of $f(z)$, so without using the integral formulas for the coefficients, if we are able to get the coefficients, we can use them to evaluate the integral $\oint_C f(z)\,dz$.

Thus, $\oint_C f(z)\,dz = 2\pi i b_1$,
(counterclockwise)

(C contains $z = z_0$, where f is singular, but no other singularities)

- The coefficient b_1 is called the residue of $f(z)$ at $z = z_0$, and is denoted by

$$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z)}$$

Example:

① $\oint_C \frac{\cos z}{z^4}\,dz$ $\quad$ C: $|z| = 1$ (counter-clockwise)

$$\frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$

$$= \frac{1}{z^4} - \frac{1}{2!\,z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$$

The coefficient of $1/z$ is 0.

$$\Rightarrow \oint_C \frac{\cos z}{z^4}\,dz = 2\pi i(0) = 0.$$

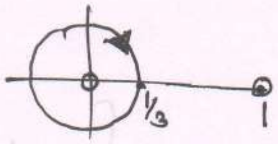
② $\oint_C \frac{\sin z}{z^4}\,dz$ $\quad$ C: $|z| = 1$.

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \cdots$$

Coefficient of $1/z$ is $-1/6$.

$$\Rightarrow \oint_C \frac{\sin z}{z^4}\,dz = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

(3) Integrate $f(z) = \frac{1}{z^3 - z^4}$ clockwise around the circle $C: |z| = 1/3$.



Ans. $\frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)}$ is singular at $z=0$ & $z=1$.

However, $z=1$ lies outside of $|z| = 1/3$. Hence we are interested in the Laurent series of $f(z)$ that converges in $0 < |z| < 1$ only. It is given by

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} \sum_{n=0}^{\infty} z^n = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots$$

Coefficient of $1/z$ is 1.

$$\Rightarrow \oint_C \frac{1}{z^3 - z^4}\, dz = \underset{\substack{\uparrow \\ \text{clockwise}}}{-} 2\pi i\,(1) = -2\pi i.$$

Remark: If we had taken the Laurent series of $\frac{1}{z^3 - z^4}$ for $|z| > 1$, we would get a wrong answer, namely, 0.

Formulas for residues at simple poles

- To calculate a residue at a pole, it may not always be feasible to write down the Laurent series and then look at the coefficient of $\frac{1}{z - z_0}$.
- It will be helpful to have direct formulas for calculating the residues.

Suppose $f(z)$ has a simple pole at $z=z_0$. Then

$$f(z)=\frac{b_1}{z-z_0}+a_0+a_1(z-z_0)+\cdots \qquad (0<|z-z_0|<R)$$

with $b_1\neq 0$.

$$\Rightarrow (z-z_0)f(z)=b_1+(z-z_0)\left(a_0+a_1(z-z_0)+\cdots\right)$$

$$\Rightarrow \lim_{z\to z_0}(z-z_0)f(z)=b_1.$$

Thus $\displaystyle\operatorname*{Res}_{z=z_0} f(z)=b_1=\lim_{z\to z_0}(z-z_0)f(z)$.

Examples

① $$\operatorname*{Res}_{z=-i}\frac{4}{1+z^2}=\lim_{z\to -i}(z+i)\frac{4}{1+z^2}=\lim_{z\to -i}\frac{(z+i).4}{(z+i)(z-i)}$$

$$=\frac{4}{-i-i}=\frac{4}{-2i}=2i.$$

Now let $f(z)=\dfrac{p(z)}{q(z)}$, $\quad$ (p, q analytic),

$p(z_0)\neq 0$ & $q(z)$ has a simple zero at z_0.

Thus f has a simple pole at z_0.

Then $$\operatorname*{Res}_{z=z_0} f(z)=\lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)}$$

$$=\lim_{z\to z_0}\frac{(z-z_0)\left(p(z_0)+(z-z_0)p'(z_0)+\cdots\right)}{\cancel{q(z_0)}^{\,0}+(z-z_0)q'(z_0)+\frac{(z-z_0)^2}{2!}q''(z_0)+\cdots}$$

$$=\lim_{z\to z_0}\frac{(z-z_0)\left(p(z_0)+(z-z_0)p'(z_0)+\cdots\right)}{(z-z_0)\left(q'(z_0)+\frac{(z-z_0)}{2!}q''(z_0)+\cdots\right)} \qquad \left(\text{Note } q'(z_0)\neq 0\right)$$

$$=\frac{p(z_0)}{q'(z_0)}.$$

Example

$$\operatorname*{Res}_{z=1/2} \tan(\pi z) = \operatorname*{Res}_{z=1/2} \frac{\sin(\pi z)}{\cos(\pi z)} = \frac{\sin(\pi/2)}{-\pi \sin(\pi/2)} = -\frac{1}{\pi}.$$

Thus, $\oint_C \tan(\pi z)\, dz = -\frac{1}{\pi} \cdot 2\pi i$, where $C: |z - \frac{1}{2}| = \frac{1}{2}$ (counter-clockwise)

Formula for the residue at a pole of any order

Suppose $f(z)$ has a pole of order m at $z = z_0$. Then,

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)}$$

Example Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at $z = 1$.

Ans. $\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz}\left((z-1)^2 f(z)\right)$

$$= \lim_{z \to 1} \frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \lim_{z \to 1} \frac{(z+2)(2z) - z^2}{(z+2)^2}$$

$$= \lim_{z \to 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{5}{9}.$$

Several singularities inside the contour. Residue Thm.

Thm. (Residue Thm.)

Let $f(z)$ be analytic inside a simple closed path C & on C, except for finitely many singular points $z_1, z_2, \ldots, z_k$ inside C. Then

$$\boxed{\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)}.$$