1/8/17 Review of complex #'s, ~~4-8AM~~ 1

Sect. 13.1 complex plane, polar forms, powers & roots

- Complex # is an ordered pair (x,y) of real #'s $z=(x,y)$. ($x \to$ Re, $y \to$ Im)
  $(x,y) \neq (y,x)$ if $x \neq y$

Let $z_1 = (x_1,y_1)$, $z_2 = (x_2, y_2)$. Then

$z_1 + z_2 = (x_1+x_2, y_1+y_2)$ & ~~$z_1 z_2 = x_1y_2 - y_1x_2$~~,

$z_1 z_2 = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$.

$(x,y) = (c,d)$ iff $x=c$ & $y=d$.

- These defns of addn & multn turn the set of all complex #'s into a field with $(0,0)$ & $(1,0)$ role of 0 & 1
- $(x,0)+(y,0) = (x+y,0)$ & $(x,0)(y,0) = (xy,0)$
- Def $i = (0,1)$.
  $i^2 = -1$. $i^2 = (0,1)(0,1) = (-1,0) = -1$.
- $a,b \in \mathbb{R} \Rightarrow (a,b) = a+bi$
- $(0,x) = ix$

Eg (Do it to do) $(4-3i)(3+7i) = (4)(3)-(-3)(7) + i((4)(7)+(-3)(7))$
$= 33+7i$.

Subtn Difference
- $z_1 - z_2 = (x_1-x_2) + i(y_1-y_2)$ is $z \ni z_1 = z + z_2$
- quotient $z = \frac{z_1}{z_2}$ $(z_2 \neq 0)$ is s.t. $z_1 = z z_2$.

$(x_1,y_1) = (x,y)(x_2,y_2) = (xx_2 - yy_2,\ xy_2 + yx_2)$

$\Rightarrow x_1 = xx_2 - yy_2 \Rightarrow$ ~~x_1~~ $x_1x_2 = xx_2^2 - yx_2y_2$
$y_1 = xy_2 + yx_2 \Rightarrow + \ y_1y_2 = xy_2^2 + yx_2y_2$

$\Rightarrow x_1x_2 + y_1y_2 = x(x_2^2+y_2^2) \Rightarrow \boxed{x = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}}$

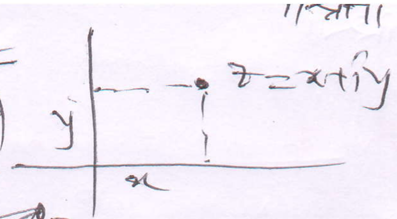
$y = \frac{xx_2 - x_1}{y_2} = \frac{\frac{x_1x_2^2 + x_2y_1y_2}{x_2^2+y_2^2} - x_1}{y_2}$

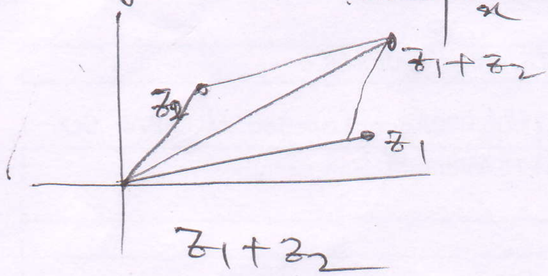
$= \frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}$

(~~Ratio~~ Multn by conjugate)

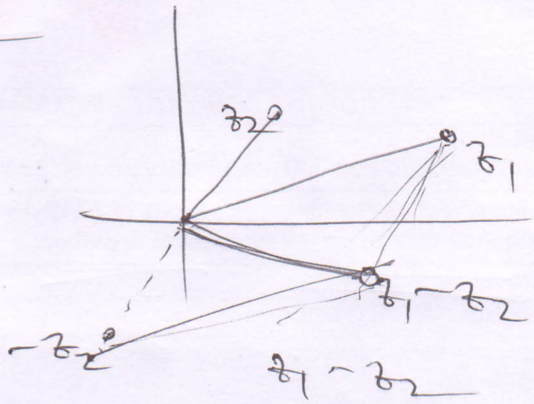
Complex plane

(Argand diag)





$z_1 + z_2$



$z_1 - z_2$

Complex conjugate

$\bar{z} = x - iy$. (refl^n of z in x-axis)

$z\bar{z} = x^2 + y^2$ is real.

$z + \bar{z} = 2x$, $z - \bar{z} = 2iy$

$Re(z) = \frac{1}{2}(z + \bar{z})$ $\quad$ $Im(z) = y = \frac{1}{2i}(z - \bar{z})$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $\quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

13.2 Find $Re\left[(1+i)^{16} z^2\right]$

$$= \frac{1}{2}\left\{(1+i)^{16} z^2 + \overline{(1+i)^{16} z^2}\right\}$$

$$= \frac{1}{2}\left((1+i)^{16} z^2 + (1-i)^{16} \bar{z}^2\right)$$

$$= \frac{1}{2}\Big(\Big(1 + \binom{16}{1} i + \binom{16}{2} i^2 + \cdots + \binom{16}{15} i^{15} + \binom{16}{16} i^{16}\Big)(x^2 - y^2 + 2ixy)$$

$$+ \Big(1 - \binom{16}{1} i + \binom{16}{2} i^2 + \cdots - \binom{16}{15} i^{15} + \binom{16}{16} i^{16}\Big)(x^2 - y^2 - 2ixy)\Big)$$

$$= \frac{1}{2}\Big(\Big(1 - \binom{16}{2} + \binom{16}{4} - \binom{16}{6} + \binom{16}{8} - \binom{16}{10} + \binom{16}{12} - \binom{16}{14} + 1\Big)$$

$$+ i\Big(\cancel{\binom{16}{1}} - \cancel{\binom{16}{3}} + \cancel{\binom{16}{5}} - \cancel{\binom{16}{7}} + \cancel{\binom{16}{9}} - \cancel{\binom{16}{11}} + \cancel{\binom{16}{13}} - \cancel{\binom{16}{15}}\Big)\Big)$$

$$(x^2 - y^2 + 2ixy)$$

$$= \left[2\Big(1 - \binom{16}{2} + \binom{16}{4} - \binom{16}{6}\Big) + \binom{16}{8}\right](x^2 - y^2)$$

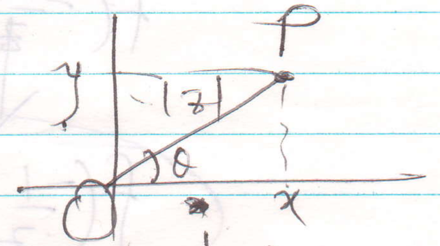
$$- 2xy\,\Big(\binom{16}{}$$

13.2 Polar forms of $\mathbb{C}$, Powers & roots

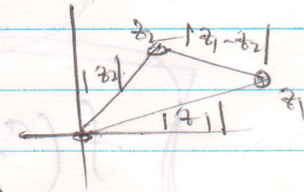
$x = r\cos\theta$, $y = r\sin\theta$; $z = r(\cos\theta + i\sin\theta)$

$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$.

(abs. val. or mod.) → dist. of z from origin

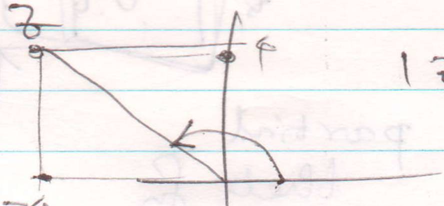


- $|z_1 - z_2| \Rightarrow$ dist. betn z_1 & z_2.



- $\tan\theta = y/x$ (θ arg. of z, denoted by $\arg z$) → directed $\angle$ from +ve x-axis.
- Angles measured in radians & +ve in a'clockwise dirn.
- For $z=0$, θ is undefined, i.e. $\arg(0)$ not defined.
- For $z \neq 0$, unique (up to $\mathbb{Z}$ mult's of 2π since cos & sin are periodic)
- For uniqueness, define princ. val. $\operatorname{Arg} z$ so that
  $-\pi < \operatorname{Arg} z \leq \pi$. → imp. for $\sqrt{\ }$'s, log's
- For real $z = x > 0$, $\operatorname{Arg} z = 0$
  For real $z = x < 0$, $\operatorname{Arg} z = \pi$.
- $\arg z = \operatorname{Arg} z \pm 2n\pi$ ($n \in \mathbb{Z}$), for $z \neq 0$.

Eg. $z = -4 + 4i$



$|z| = \sqrt{(-4)^2 + (4)^2} = \sqrt{32} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

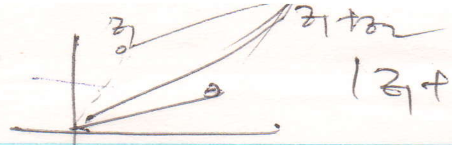
$\Rightarrow z = 4\sqrt{2}\left(\cos\left(\frac{3\pi}{4} + 2n\pi\right) + i\sin\left(\frac{3\pi}{4} + 2n\pi\right)\right)$

Note $\tan\theta$ has period π

- if $\theta_1 = \arg(1+i)$, $\theta_2 = \arg(-1, -i)$
  $\tan\theta_1 = \tan\theta_2 = 1$

$\triangle$ ineq. $|z_1 + z_2| \leq |z_1| + |z_2|$,

No natural way of ordering complex numbers



$|z_1+\dots+z_n| \le |z_1|+\dots+|z_n|$

Multⁿ & Divⁿ in polar form

- $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
- $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ (Prove!)

$\Rightarrow |z_1 z_2| = r_1 r_2 = |z_1||z_2|$.

- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ (up to mult's of 2π)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$ since $z_1 = \dfrac{z_1}{z_2}\cdot z_2 \Rightarrow |z_1| = \left|\dfrac{z_1}{z_2}\right||z_2|$
- $\arg z_1 = \arg\left(\dfrac{z_1}{z_2}\cdot z_2\right) = \arg\dfrac{z_1}{z_2} + \arg z_2$. (up to mult's of 2π)

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

Eg

$z_1 = -2+2i$, $z_2 = 3i$

$z_1 z_2 = -6-6i$, $\quad \dfrac{z_1}{z_2} = \dfrac{(-2+2i)(-3i)}{(3i)(-3i)}$

$= \dfrac{6+6i}{9} = \dfrac{2}{3} + \dfrac{2}{3}i$.

$|z_1 z_2| = \underbrace{\sqrt{(-2)^2+2^2}\sqrt{3^2}}_{|z_1||z_2|} = 3\cdot 2\sqrt{2} = 6\sqrt{2}$,

$\left|\dfrac{z_1}{z_2}\right| = \sqrt{\left(\dfrac{2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^2} = \dfrac{2}{3}\sqrt{2}$.

$\dfrac{|z_1|}{|z_2|} = \dfrac{2\sqrt{2}}{3}$,

$\operatorname{Arg} z_1 = \dfrac{3\pi}{4}$ $\quad \operatorname{Arg} z_2 = \dfrac{\pi}{2}$ $\quad \operatorname{Arg}(z_1 z_2) = \dfrac{5\pi}{4}$

$\operatorname{Arg}\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{4}$.