MA 502 : SEM. II (Part 2) - Lecture 1 8/6/2020

Already covered : Maximum modulus theorem

If G is a region and $f: G \to \mathbb{C}$ is an analytic function such that there is a point a in G with $|f(a)| \geq |f(z)|$ $\forall z \in G$, then f is constant.

Proof: Let $\bar{B}(a, r) \subseteq G$ for some $r > 0$, and let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$.

Now $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} \cdot rie^{it}\,dt$

$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$

$\Rightarrow |f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\,dt = |f(a)|$

(by the hypothesis)

$\Rightarrow$

$\frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|\,dt = |f(a)|$, so that $\int_0^{2\pi} \left(|f(a)| - |f(a+re^{it})|\right)dt = 0$.

However, by the hypothesis again, the integral is non-negative ; hence $|f(a)| = |f(a+re^{it})|$ $\forall t$.

Since r was arbitrary, we see that f maps any disk $B(a;R) \subseteq G$ into the circle $|z| = |f(a)|$.

$\Rightarrow f$ is constant on $B(a;R)$ & hence $f(z) =$ constant for $\forall z \in G$ (by identity theorem.) (in fact, $f(a)$) ■

Other versions of the Maximum modulus theorem

SECOND VERSION - Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on $\bar{G}$ which is analytic in G. Then

$$\max\{|f(z)| : z \in \bar{G}\} = \max\{|f(z)| : z \in \partial G\}$$

Proof: Since G is bounded, so is $\bar{G}$. Along with the fact that $\bar{G}$ is closed, we see that $\bar{G}$ is compact. (Closed and bounded sets of $\mathbb{C}$ are compact.)

Next, a continuous function on a compact set is bounded. Since f is continuous, $|f|$ is continuous.

Thus $\exists\ a \in \bar{G} \ni |f(z)| \leq |f(a)| \ \forall\ z \in \bar{G}$. — (1)

Case 1: f is a constant function on $\bar{G}$: Then, of course,

$$\max\{|f(z)| : z \in \bar{G}\} = \max\{|f(z)| : z \in \partial G\}.$$

Case 2: f is not constant. Then by the original version, $\nexists\ b \in G \ni |f(z)| \leq |f(b)| \ \forall\ z \in G$. — (2)

Thus, from (1) and (2),

$$\max\{|f(z)| : z \in \bar{G}\} = \max\{|f(z)| : z \in \partial G\}.$$

Remarks: (1) In the first version, we need connectedness. But that is not the requirement in the second because even if A and B are connected sets that form a separation of G, we can apply version 1 to each of A and B to ~~complete~~ reach the conclusion of version 2 for A, and for B as well. Then combining the 2 results, we get the same conclusion for G, i.e.,

$$\max\{|f(z)| : z \in \bar{G}\} = \max\{|f(z)| : z \in \partial G\}.$$

(2) Let $G = \{x+iy : -\frac{\pi}{2} < y < \frac{\pi}{2}\}$. Let $f(z) = e^{e^z}$. Then f is continuous on $\bar{G}$ as well as analytic on G. If $z \in \partial G$, then $z = x \pm \frac{\pi i}{2}$, so

$$|f(z)| = |\exp(e^{x \pm \frac{i\pi}{2}})| = |\exp(\pm i e^{x})| = 1$$

However, $\lim\limits_{\substack{x \to \infty \\ x \in \mathbb{R}}} f(x) = \infty$. Does this contradict the M.M.P.?

Definitions Let $f: G \to \mathbb{R}$ and $a \in \bar{G}$ or $a = \infty$. Then,

- $\limsup_{z \to a} f(z) = \lim_{r \to 0^+} \sup\{f(z) : z \in G \cap B(a;r)\}$.

(If $a = \infty$, $B(a;r)$ is the ball in the metric of $\mathbb{C}_\infty$.)

- $\liminf_{z \to a} f(z) = \lim_{r \to 0^+} \inf\{f(z) : z \in G \cap B(a;r)\}$.
- $\lim_{z \to a} f(z) = \alpha$ iff $\limsup_{z \to a} f(z) = \liminf_{z \to a} f(z) = \alpha$.
- Let $G \subset \mathbb{C}$. Let $\partial_\infty G$ denote the boundary of G in $\mathbb{C}_\infty$, the extended boundary of G.

$\partial_\infty G = \partial G$ if G is bounded, and

$\partial_\infty G = \partial G \cup \infty$ if G is unbounded.

Maximum modulus theorem – Third version

Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\limsup_{z \to a} |f(z)| \le M$ $\forall a \in \partial_\infty G$. Then $|f(z)| \le M$ $\forall z \in G$.

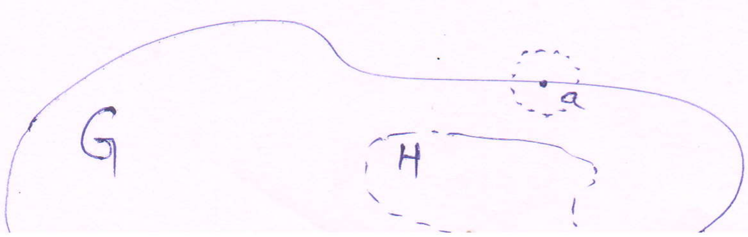
Proof: Let $\delta > 0$ be an arbitrary positive real number & let $H := \{z \in G : |f(z)| > M + \delta\}$.

Goal: $H = \phi$.

First, f continuous $\Rightarrow |f|$ is continuous.

Since the pre-image of an open set under a continuous map is open, we have H to be an open set.

Now $\limsup_{z \to a} |f(z)| \le M$ $\forall a \in \partial_\infty G$ implies that $\exists\, r > 0 \ni$ $|f(z)| < M + \delta$ $\forall z \in G \cap B(a, r)$. —— (1)



Now if z is a limit point of a sequence $\{z_n\}$ in H.
Then $|f|$ continuous implies $|f(z)| = \lim_{n\to\infty} |f(z_n)| > M+\delta$.
But then $z \notin \partial_\infty G$.
Hence $\bar{H} \subseteq G$. —— (2)
Now (1) holds even when G is unbounded and $a = \infty$.
(If G is bounded, (2) (or just the fact $H \subseteq G$) implies H is bounded.
If G is unbounded, $\exists$ a neighborhood N of $a = \infty \ni$ for $z \in G \cap N$, $|f(z)| < M+\delta$. Then (2) implies H is bounded)
But H bounded implies $\bar{H}$ is bounded, and being closed as well, we see that $\bar{H}$ is compact.
By the second version of the Maximum modulus theorem,
$\max\{|f(z)| : z \in \bar{H}\} = \max\{|f(z)| : z \in \partial H\}$
But for $z \in \partial H$, note that $|f(z)| = M+\delta$ since
$\bar{H} \subset \{z : |f(z)| \geq M+\delta\}$.
This must mean $H = \phi$ or f is a constant. That f could be a constant is clear. If not, note that
$\max\{|f(z)| : z \in \bar{H}\} = M+\delta$.
So $\nexists\, z \in G \ni |f(z)| > M+\delta$. $\Rightarrow H = \phi$.
Now if f is constant, say A, clearly, $A = \limsup_{z\to a} |f(z)| \leq M$
$\Rightarrow H = \phi$ in this case as well.
$\Rightarrow |f(z)| \leq M \quad \forall z \in G$.

<u>Remark</u>: Let $G = \{z : |\text{Im}(z)| < \pi/2\}$, $f(z) = \exp(e^z)$.
Note that for all $a \in \partial G$, $\limsup_{z\to a} |f(z)| \leq 1$, but not for $a = \infty$. Thus f is <u>not</u> bounded on G.

The index of a closed curve

Note that if $\gamma(t) = a + e^{int}$ $(0 \le t \le 2\pi)$, then

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{in\, e^{int}}{e^{int}}\, dt = 2\pi i n.$$

However, this evaluation is not dependent on the path γ above.

Thm. 5.1 If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer.

Proof: We prove this only in the case when γ is smooth. Then let $g : [0,1] \to \mathbb{C}$ be defined by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds.$$

Obviously, $g(0) = 0$ & $g(1) = \int_0^1 \frac{\gamma'(s)\, ds}{\gamma(s)-a} = \int_\gamma \frac{dz}{z-a}$

Also, $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$ for $0 \le t \le 1$.

But then

$$\frac{d}{dt} e^{-g(t)}(\gamma(t)-a) = e^{-g(t)}\gamma'(t) - g'(t)e^{-g(t)}(\gamma(t)-a)$$

$$= e^{-g(t)}\{\gamma'(t) - \gamma'(t)\} = 0.$$

$\Rightarrow e^{-g(t)}(\gamma(t)-a)$ is a constant function given by $e^{-g(0)}(\gamma(0)-a) = \gamma(0) - a = e^{-g(1)}(\gamma(1)-a)$.

Since γ is closed, $\gamma(0) = \gamma(1)$.

$\Rightarrow e^{-g(1)} = 1 \Rightarrow g(1) = 2\pi i k$ for some integer k.

$\square$