- A non-constant entire fn., being unbounded, has a pole if it's a polynomial or an essential singularity if it is not.

- An analytic fn. whose only singularities in the finite plane are poles is called <u>meromorphic fn</u>, e.g. cosec(z),

Sect. 16.3 - Residue Integration Method

- If f is analytic on and inside a simple closed path C, then $\oint_C f(z)dz = 0$ by Cauchy's integral theorem.

- If f(z) has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C & inside it, f(z) admits a Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

which converges in $0 < |z - z_0| < R$ for some positive R.

$\boxed{\text{MAIN IDEA}}$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz .$$

Now there are many ways to obtain Laurent series of f(z), so without using the integral formulas for the coefficients, if we are able to get the coefficients, we can use them to evaluate the integral $\oint_C f(z)\,dz$.

Thus, $\oint_C f(z)\,dz = 2\pi i b_1$,
(counterclockwise)

(C contains $z = z_0$, where f is singular, but no other singularities)

- The coefficient b_1 is called the residue of $f(z)$ at $z = z_0$, and is denoted by

$$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z)}$$

Example:

① $\oint_C \frac{\cos z}{z^4}\,dz$ C: $|z| = 1$
(counter-clockwise)

$$\frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$

$$= \frac{1}{z^4} - \frac{1}{2!\,z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$$

The coefficient of $1/z$ is 0.

$$\Rightarrow \oint_C \frac{\cos z}{z^4}\,dz = 2\pi i(0) = 0.$$

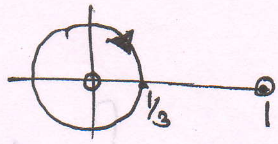
② $\oint_C \frac{\sin z}{z^4}\,dz$ C: $|z| = 1$.

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \cdots$$

Coefficient of $1/z$ is $-1/6$

$$\Rightarrow \oint_C \frac{\sin z}{z^4}\,dz = 2\pi i(-1/6) = -\frac{\pi i}{3}.$$

(3) Integrate $f(z)=\frac{1}{z^3-z^4}$ clockwise around the circle $C: |z| = 1/3$.



Ans. $\frac{1}{z^3-z^4} = \frac{1}{z^3(1-z)}$ is singular at $z=0$ & $z=1$.

However, $z=1$ lies outside of $|z|=1/3$. Hence we are interested in the Laurent series of $f(z)$ that converges in $0<|z|<1$ only. It is given by

$$\frac{1}{z^3-z^4} = \frac{1}{z^3}\sum_{n=0}^{\infty} z^n = \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\dots$$

Coefficient of $1/z$ is 1.

$$\Rightarrow \oint_C \frac{1}{z^3-z^4}\,dz = \underset{\substack{\uparrow \\ \text{clockwise}}}{-}2\pi i\,(1) = -2\pi i.$$

Remark: If we had taken the Laurent series of $\frac{1}{z^3-z^4}$ for $|z|>1$, we would get a wrong answer, namely, 0.

Formulas for residues at simple poles

- To calculate a residue at a pole, it may not always be feasible to write down the Laurent series and then look at the coefficient of $\frac{1}{z-z_0}$.
- It will be helpful to have direct formulas for calculating the residues.

Suppose $f(z)$ has a simple pole at $z = z_0$. Then

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots \qquad (0 < |z - z_0| < R)$$

with $b_1 \neq 0$.

$$\Rightarrow (z - z_0) f(z) = b_1 + (z - z_0)(a_0 + a_1(z - z_0) + \cdots)$$

$$\Rightarrow \lim_{z \to z_0} (z - z_0) f(z) = b_1 .$$

Thus $\displaystyle \operatorname*{Res}_{z = z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$.

Examples

① $$\operatorname*{Res}_{z=-i} \frac{4}{1+z^2} = \lim_{z \to -i} (z+i) \frac{4}{1+z^2} = \lim_{z \to -i} \frac{(z+i).4}{(z+i)(z-i)}$$

$$= \frac{4}{-i-i} = \frac{4}{-2i} = 2i .$$

Now let $f(z) = \dfrac{p(z)}{q(z)}$, ($p, q$ analytic),

$p(z_0) \neq 0$ & $q(z)$ has a simple zero at z_0.

Thus f has a simple pole at z_0.

Then $$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)}$$

$$= \lim_{z \to z_0} \frac{(z - z_0)(p(z_0) + (z - z_0) p'(z_0) + \cdots)}{\cancelto{0}{q(z_0)} + (z - z_0) q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots}$$

$$= \lim_{z \to z_0} \frac{(z - z_0)(p(z_0) + (z - z_0) p'(z_0) + \cdots)}{(z - z_0)(q'(z_0) + \frac{(z - z_0)}{2!} q''(z_0) + \cdots)} \qquad (\text{Note } q'(z_0) \neq 0)$$

$$= \frac{p(z_0)}{q'(z_0)} .$$

Example

$$\operatorname*{Res}_{z=1/2} \tan(\pi z) = \operatorname*{Res}_{z=1/2} \frac{\sin(\pi z)}{\cos(\pi z)} = \frac{\sin(\pi/2)}{-\pi \sin(\pi/2)} = -\frac{1}{\pi}.$$

Thus, $\oint_C \tan(\pi z)\,dz = -\frac{1}{\pi} \cdot 2\pi i$, where $C: |z - \frac{1}{2}| = \frac{1}{2}$ (counter-clockwise)

Formula for the residue at a pole of any order

Suppose $f(z)$ has a pole of order m at $z = z_0$. Then,

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)}$$

Example Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at $z = 1$.

Ans. $$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 f(z)\right)$$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \lim_{z\to 1} \frac{(z+2)(2z) - z^2}{(z+2)^2}$$

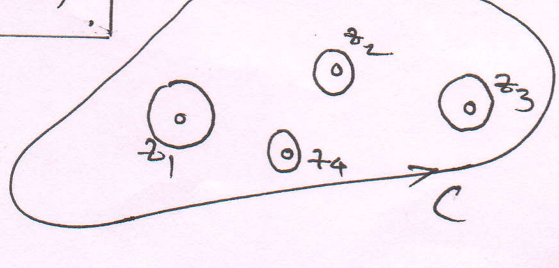
$$= \lim_{z\to 1} \frac{z^2+4z}{(z+2)^2} = \frac{5}{9}.$$

Several singularities inside the contour. Residue Thm.

Thm. (Residue Thm.)

Let $f(z)$ be analytic inside a simple closed path C & on C, except for finitely many singular points $z_1, z_2, \ldots, z_k$ inside C. Then

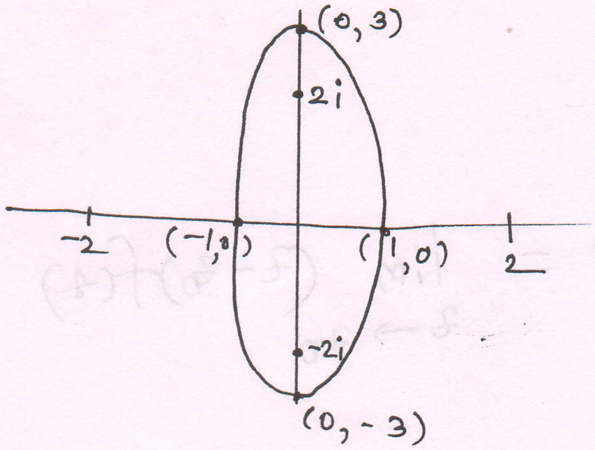
$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)}$$



Example

Evaluate the integral $\oint_C \left(\frac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z} \right) dz$,

where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise)



$$\oint_C \left(\frac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z} \right) dz = \oint_C \frac{z e^{\pi z}}{z^4 - 16} dz + \oint_C z e^{\pi/z} dz$$

$$=: \quad 1 + 2$$

By residue theorem,

$$1 = 2\pi i \left(\operatorname*{Res}_{z=2i} \frac{z e^{\pi z}}{z^4 - 16} + \operatorname*{Res}_{z=-2i} \frac{z e^{\pi z}}{z^4 - 16} \right) \quad \left[\begin{array}{c} \text{Simple} \\ \text{poles} \\ \text{at } z = \pm 2i \end{array} \right]$$

$$= 2\pi i \left\{ \left. \frac{z e^{\pi z}}{4 z^3} \right|_{z=2i} + \left. \frac{z e^{\pi z}}{4 z^3} \right|_{z=-2i} \right\}$$

$$= 2\pi i \left(-\frac{1}{16} - \frac{1}{16} \right) = -\frac{\pi i}{4}.$$

2 : $z e^{\pi/z}$ has an essential singularity at $z=0$.

$$z e^{\pi/z} = z \left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!\, z^2} + \cdots \right) = z + \pi + \frac{\pi^2}{2z} + \cdots$$

$$\Rightarrow \quad \oint_C z e^{\pi z} dz = 2\pi i \left(\frac{\pi^2}{2} \right) = \pi^3 i$$

Thus $\boxed{\text{Ans. } \pi^3 i - \frac{\pi i}{4} = \pi i \left(\pi^2 - \frac{1}{4} \right).}$

Sect. 16.4 – Evaluation of real integrals

- Residue integration is useful for evaluating certain REAL integrals!

(1) Integrals of rational functions of $\cos\theta$ and $\sin\theta$

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$F(\cos\theta, \sin\theta)$: real rational function of $\cos\theta$ & $\sin\theta$

Method: (i) Let $e^{i\theta} = z$ and set

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

(ii) Then F becomes a rational function in z.

(iii) $\frac{dz}{d\theta} = ie^{i\theta} = iz \Rightarrow d\theta = \frac{dz}{iz}$.

Hence $I = \oint_C f(z)\frac{dz}{iz}$, where C: unit circle traversed counterclockwise

Example: Evaluate $\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\, d\theta$. Denote the integral by I.

Let $z = e^{i\theta}$ $\quad \frac{dz}{iz} = d\theta$

Hence $I = \oint_C \frac{\frac{1}{2i}\left(z-\frac{1}{z}\right)}{3+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$ $\quad$ (C: unit circle (counter-clockwise)

$$= -\oint_C \frac{z^2-1}{z(z^2+6z+1)}\, dz$$

$$= -\oint_C \frac{(z^2-1)}{z(z+3-2\sqrt{2})(z+3+2\sqrt{2})}\,dz.$$

Out of the three poles $z=0$, $z=-3+2\sqrt{2}$, $z=-3-2\sqrt{2}$, two of them, namely 0 & $-3+2\sqrt{2}$ lie within the unit circle. Hence by residue theorem,

$$I = -2\pi i\left\{\operatorname*{Res}_{z=0} \frac{z^2-1}{z(z^2+6z+1)} + \operatorname*{Res}_{z=-3+2\sqrt{2}} \frac{z^2-1}{z(z^2+6z+1)}\right\}$$

$$= -2\pi i\left\{\frac{0^2-1}{0^2+6(0)+1} + \frac{(-3+2\sqrt{2})^2-1}{(-3+2\sqrt{2})(-\cancel{3}+2\sqrt{2}+\cancel{3}+2\sqrt{2})}\right\}$$

$$= -2\pi i\left\{-1 + \frac{16-12\sqrt{2}}{16-12\sqrt{2}}\right\} = 0.$$

(2) Improper integrals of rational functions

$$I = \int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$$

If both limits exist, we couple the two independent passages to $-\infty$ & ∞, and write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to+\infty}\int_{-R}^{R} f(x)\,dx.$$

Conditions of f:

(1) f is a real rational function whose denominator is different from zero for all real x, that is, f does not have any poles on the real axis.

(2) $\deg(\text{Denominator}) \geq \deg(\text{Numerator}) + 2$, that is, $\deg(f(x)) \leq -2$.