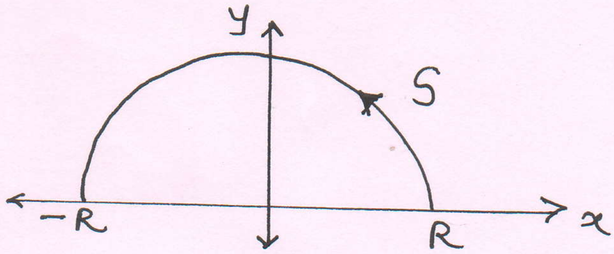
Consider the corresponding contour integral $\oint_C f(z)\,dz$ around a path C shown below.



$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z),$$

where R is chosen so large that it encloses all of the poles of f in the upper-half plane (UHP).

$$\Rightarrow \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res}(f(z)) - \int_S f(z)\,dz.$$

Claim: $\lim_{R\to\infty} \int_S f(z)\,dz = 0$.

Proof: Since $\deg(f(x)) \le -2$, for sufficiently large constants k and R_0,

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

$\Rightarrow$ By the ML-inequality,

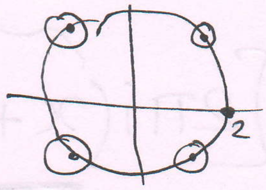
$$\left|\int_S f(z)\,dz\right| < \frac{k}{R^2}\cdot \pi R = \frac{\pi k}{R} \qquad (R > R_0)$$

$$\Rightarrow \lim_{R\to\infty} \int_S f(z)\,dz = 0.$$

Thus,
$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)},$$

where we sum over all the residues of $f(z)$ corresponding to the poles of $f(z)$ in the UHP.

Example Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+16}$. Let $f(z) = \frac{1}{z^4+16}$



- $\text{Deg}(f(x)) = -4 < -2$.
- No poles of the real axis.

Hence $\int_{-\infty}^{\infty} \frac{dx}{x^4+16} = 2\pi i \sum \text{Res}\, f(z)$, where the sum is over all poles of f in the UHP.

Poles of f are at z where $z^4+16=0$, i.e., $z^4=-16$, Thus they are $2e^{i\pi/4}$, $2e^{i3\pi/4}$, $2e^{-i3\pi/4}$ and $2e^{-i\pi/4}$.

Out of them, the first two lie in the UHP.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+16} = 2\pi i \left\{ \operatorname*{Res}_{z=2e^{i\pi/4}} f(z) + \operatorname*{Res}_{z=2e^{i3\pi/4}} f(z) \right\}$$

$$= 2\pi i \left\{ \frac{1}{4z^3}\Big|_{z=2e^{i\pi/4}} + \frac{1}{4z^3}\Big|_{z=2e^{i3\pi/4}} \right\}$$

$$= 2\pi i \left\{ \frac{1}{4\times(2e^{i\pi/4})^3} + \frac{1}{4(2e^{i3\pi/4})^3} \right\}$$

$$= \frac{2\pi i}{32}\left\{ e^{-3i\pi/4} + e^{-i\pi/4} \right\} = \frac{2\pi i}{32}\left\{ e^{-\pi i + i\pi/4} + e^{-i\pi/4} \right\}$$

$$= -\frac{2\pi i}{32}\left\{ e^{i\pi/4} - e^{-i\pi/4} \right\} = -\frac{2\pi i}{32} \cdot 2i\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{4\pi}{32} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{8\sqrt{2}}.$$

Consider the improper integral $\int_A^B f(x)\,dx$ s.t. $\lim\limits_{\substack{x\to\alpha \\ (A<\alpha<B)}} |f(x)| = \infty$. By defn;

$$\int_A^B f(x)\,dx = \lim_{\varepsilon\to 0}\int_A^{\alpha-\varepsilon} f(x)\,dx + \lim_{\eta\to 0}\int_{\alpha+\eta}^{B} f(x)\,dx,$$

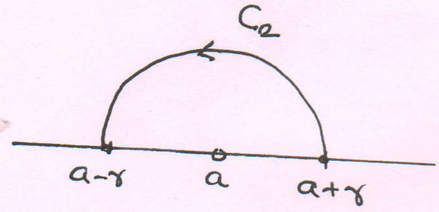
where both ε & η approach zero independently and through positive values.

It may be that neither limit exists if ε & $\eta \to 0$ independently, however, $\lim\limits_{\varepsilon\to 0}\left(\int_A^{\alpha-\varepsilon} f(x)\,dx + \int_{\alpha+\varepsilon}^{B} f(x)\,dx\right)$ exists. This is called the <u>Cauchy principal value</u> of the integral and is written $\text{P.V.}\int_A^B f(x)\,dx$.

Eg. $$\text{P.V.}\int_{-1}^{1}\frac{dx}{x^3} = \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon}\frac{dx}{x^3} + \int_{\varepsilon}^{1}\frac{dx}{x^3}\right] = 0.$$

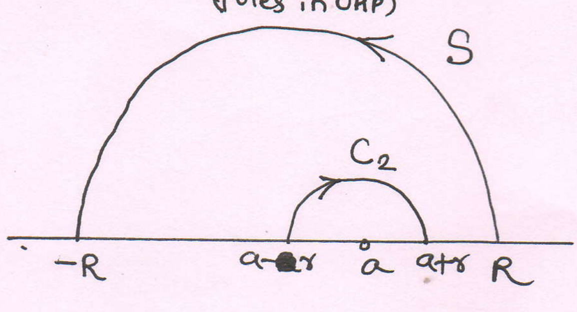
<u>Thm.</u> <u>(Simple poles on the real axis)</u> Let $\deg(f(x)) \le -2$.

(i) If $f(z)$ has a <u>simple</u> pole $z=a$ on the real axis, then



$$\lim_{r\to 0}\int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

(ii) $$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\text{(poles in UHP)}} \text{Res}\, f(z) + \pi i \sum_{\substack{\text{(poles} \\ \text{on real} \\ \text{axis)}}} \text{Res}\, f(z)$$



Example Find the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{x}{8-x^3}\,dx \quad . \quad \text{Let } f(x) = \frac{x}{8-x^3}.$$

- $\deg(f(x)) \leq -2.$ ✓
- Poles at $x \ni 8 - x^3 = 0$, i.e. $(2-x)(x^2+2x+4) = 0$,

  i.e. at $x = \underset{\in\mathbb{R}}{\underset{\downarrow}{2}},\ \underset{(\text{UHP})}{\underset{\downarrow}{-1+\sqrt{3}i}},\ -1-\sqrt{3}i \;.\; \Longleftarrow$ Simple poles

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x}{8-x^3}\,dx = 2\pi i \operatorname*{Res}_{z=-1+\sqrt{3}i} f(z) + \pi i \operatorname*{Res}_{z=2} f(z).$$

- $$\operatorname*{Res}_{z=-1+\sqrt{3}i} f(z) = \lim_{z\to(-1+\sqrt{3}i)} \frac{(z-(-1+\sqrt{3}i))\,z}{(2-z)(z-(-1-\sqrt{3}i))\,(z-(-1+\sqrt{3}i))}$$

$$= \frac{-1+\sqrt{3}i}{(3-\sqrt{3}i)(2\sqrt{3}i)} = \frac{-1+\sqrt{3}i}{(\sqrt{3}+3i)\cdot 2\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} \; \frac{(-1+\sqrt{3}i)(\sqrt{3}-3i)}{3+9}$$

$$= \frac{1}{24\sqrt{3}} \left(-\sqrt{3} + 3i + 3i + 3\sqrt{3}\right)$$

$$= \frac{2\sqrt{3}+6i}{24\sqrt{3}} = \frac{1+\sqrt{3}i}{12}.$$

(In a simpler way, $\displaystyle \operatorname*{Res}_{z=-1+\sqrt{3}i} f(z) = \left.\frac{z}{-3z^2}\right|_{z=-1+\sqrt{3}i} = \frac{1+\sqrt{3}i}{12}$

$\displaystyle \operatorname*{Res}_{z=2} f(z) = \left.\frac{z}{-3z^2}\right|_{z=2} = -\frac{1}{6}.$)

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty} \frac{x\,dx}{8-x^3} = 2\pi i\left(\frac{1+\sqrt{3}i}{12}\right) - \frac{\pi i}{6} = -\frac{\pi}{2\sqrt{3}}.$$

Fourier integrals

To evaluate : $\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx$, $\int_{-\infty}^{\infty} f(x)\sin(sx)\,dx$ (s real & positive)

- $\deg(f(x)) \leq -2$.

Thm.
$$\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx = -2\pi \sum \operatorname{Im}(\operatorname{Res}[f(z)e^{isz}])$$
$$\int_{-\infty}^{\infty} f(x)\sin(sx)\,dx = 2\pi \sum \operatorname{Re}(\operatorname{Res}[f(z)e^{isz}]).$$

Example

Prove that
$$\int_{-\infty}^{\infty} \frac{\cos(sx)}{k^2+x^2}\,dx = \frac{\pi}{k} e^{-ks}$$
$$\int_{-\infty}^{\infty} \frac{\sin(sx)\,dx}{k^2+x^2} = 0 \qquad (s>0, k>0).$$

Proof :- ~~Let~~ ~~f(z)~~ Consider $\frac{e^{isz}}{k^2+z^2}$. It has only one pole, namely $z=ik$, in the UHP.

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2+z^2} = \left.\frac{e^{isz}}{2z}\right|_{z=ik} = \frac{e^{-ks}}{2ik}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{isx}}{k^2+x^2}\,dx = 2\pi i \cdot \frac{e^{-ks}}{2ik} = \frac{\pi e^{-ks}}{k}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(sx)}{k^2+x^2}\,dx = \frac{\pi e^{-ks}}{k}, \quad \int_{-\infty}^{\infty} \frac{\sin(sx)\,dx}{k^2+x^2} = 0.$$

Tutorial 6 – Problem 8

Prove that if a non-constant polynomial $p(z)$ with complex coefficients has all of its roots in the half-plane $Re(z) > 0$, then all of the roots of its derivative $p'(z)$ are also in the right-half plane $Re(z) > 0$.

Solution: By the Fundamental Theorem of Algebra, we know that every non-constant polynomial has roots in $\mathbb{C}$. So let $p(z)$ be factored as

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \quad (\text{assuming } \deg(p(z)) = n)$$

Note that $z_i = a_i + b_i$ satisfies $a_i > 0 \ \forall\, i \ni 1 \le i \le n$.

We prove the above result by contradiction.

Assume that $\exists$ a zero of $p'(z)$, say, $z_0 = a_0 + ib_0$, such that $a_0 \le 0$. Then $p(z_0) \ne 0$ by the hypothesis.

Hence $\dfrac{p'(z_0)}{p(z_0)} = 0$.

But $p(z_0) = (z_0 - z_1)(z_0 - z_2) \cdots (z_0 - z_n)$ implies

$\log p(z_0) = \log(z_0 - z_1) + \cdots + \log(z_0 - z_n)$ so that

$$\frac{p'(z_0)}{p(z_0)} = \sum_{i=1}^{n} \frac{1}{z_0 - z_i}.$$

But $\displaystyle\sum_{i=1}^{n} \frac{1}{z_0 - z_i} = \sum_{i=1}^{n} \frac{\overline{z_0} - \overline{z_i}}{|z_0 - z_i|^2}$

Hence $\displaystyle\sum_{i=1}^{n} \frac{\overline{z_0} - \overline{z_i}}{|z_0 - z_i|^2} = 0 \implies \sum_{i=1}^{n} \frac{z_0 - z_i}{|z_0 - z_i|^2} = 0$

$$\Rightarrow \sum_{i=1}^{n} \frac{(a_0 - a_i) + i(b_0 - b_i)}{|z_0 - z_i|^2} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{a_0 - a_i}{|z_0 - z_i|^2} = 0$$

But $a_0 - a_i < 0 \ \forall \ i$

This gives the contradiction.

Hence all of the roots of $p'(z)$ lies in $Re(z) > 0$.

$\blacksquare$