- Let z_1 and z_2 be 2 complex numbers.

Then, $|z_1 z_2| = |z_1||z_2|$ —— (1)

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ —— (2)

(up to multiples of 2π)

Hence by induction, if $z = r(\cos\theta + i\sin\theta)$,

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)} \text{ for } n = 0, 1, 2, \dots \quad \text{—— (3)}$$

- If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

So let $z_1 = 1$ and $z_2 = z^n$. Then from (3),

$$z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta)).$$

This gives (3) for all integers n.

- Let $|z| = r = 1$ in (3). Then

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.}$$

This is known as <u>De Moivre's formula</u>.

- Prove
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  - $\sin 2\theta = 2\sin\theta\cos\theta$

Roots

Given $z = w^n$, $n \in \mathbb{N}$, to a given $z \neq 0$, there correspond exactly n distinct values of w, known as n^{th} roots of z, written as

$$w = \sqrt[n]{z} = z^{1/n}.$$

- Hence $z^{1/n}$ is a multi-valued function, in fact, n-valued.

- <u>Obtaining the n^{th} roots of z</u>

Let $z = r(\cos\theta + i\sin\theta)$

$w = R(\cos\phi + i\sin\phi)$.

Then $w^n = z$ implies

$$R^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

$\Rightarrow$ $R^n = r$, and hence $R = \sqrt[n]{r} > 0$ (Why?)

Also, $n\phi = \theta + 2k\pi$, $k \in \mathbb{Z}$

$$\Rightarrow \phi = \frac{\theta}{n} + \frac{2k\pi}{n}.$$

- $k = 0, 1, 2, \ldots, n-1$ give distinct values of w.
- Note that if $k = n$, then $\frac{2k\pi}{n} = 2\pi$.

By the periodicity of sine and cosine,

$\cos\phi = \cos\left(\frac{\theta}{n} + 2\pi\right) = \cos\left(\frac{\theta}{n}\right)$ & $\sin\phi = \sin\left(\frac{\theta}{n}\right)$

Hence, $\boxed{\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)}$

- For $z=1$, $|z| = r = 1$, hence $\operatorname{Arg} z = 0$ implies

$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \; k = 0,1,2,\ldots,n-1.$$

(n^{th} Roots of unity)

- Lie on the unit circle $|z| = 1$.

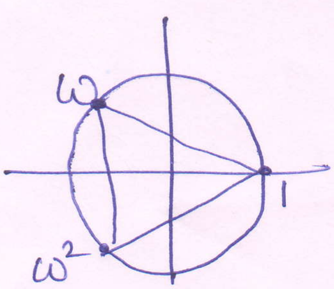
e.g. $\sqrt[3]{1} = 1, \; \frac{-1}{2} \pm \frac{\sqrt{3}\,i}{2}$.

- Let ω denote the n^{th} root of unity corresponding to $k=1$. Then the n^{th} roots of unity can be written in terms of ω as

$$1, \omega, \omega^2, \ldots, \omega^{n-1}.$$

- If $z \in \mathbb{C}$ and $w_1 = \sqrt[n]{z}$, then these n^{th} roots of z are given by

$$w_1, w_1\omega, w_1\omega^2, \ldots, w_1\omega^{n-1}$$

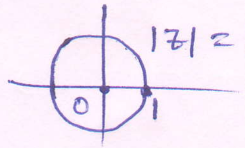


$$\omega = \frac{-1}{2} + \frac{\sqrt{3}\,i}{2}.$$

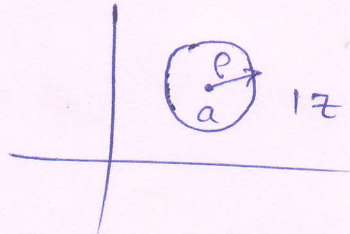
Cube-roots of unity.

Section 13.3

- $|z|=1$ (Unit circle) is the set of all complex numbers which are at distance 1 from the origin.

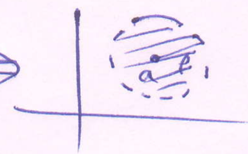


- Similarly, $|z-a|=\rho$ is the set of all z whose distance $|z-a|$ from the center equals ρ, equivalently, the set of all z whose distance from a equals ρ.

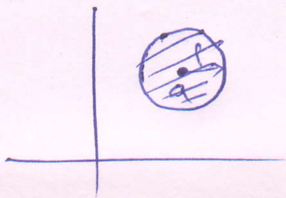


$|z-a|=\rho$

- Open circular disk $|z-a|<\rho \Rightarrow$

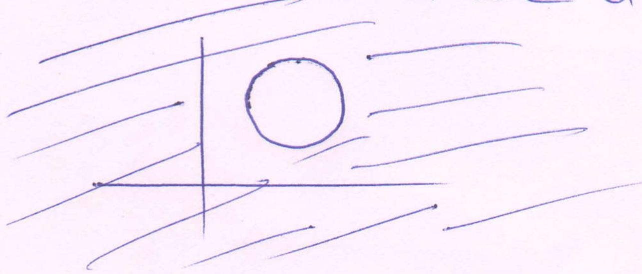


- closed circular disk



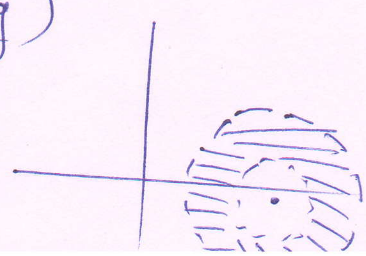
$|z-a| \leq \rho$

- exterior of a closed circular disk
  $|z-a|>\rho$



- open-circular disk is a neighborhood of a.
  $(|z-a|<\rho,\ \rho>0)$

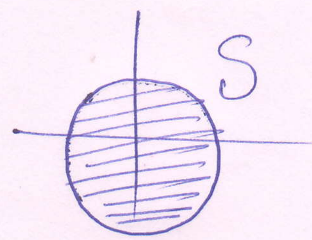
- Open annulus (circular ring)
  $\rho_1<|z-a|<\rho_2$



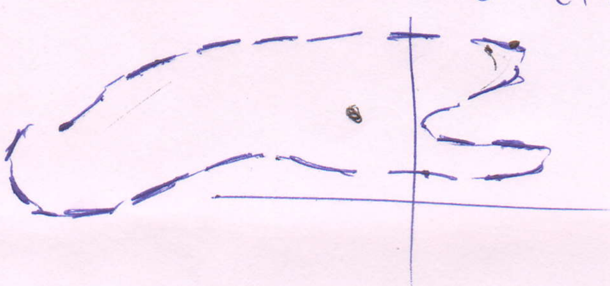
- Closed annulus ρ_1 ...

- Upper Half plane (UHP) : $\{z = x+iy \mid y>0\}$
- Lower Half plane (LHP) : $\{z = x+iy \mid y<0\}$
- Right Half plane (RHP) : $\{z = x+iy \mid x>0\}$
- Left Half plane (LHP) : $\{z = x+iy \mid x<0\}$.
- Set S is <u>open</u> if every point of S has a neighborhood (nbhd) consisting entirely of points belonging to S.

  — Is this an open set ?

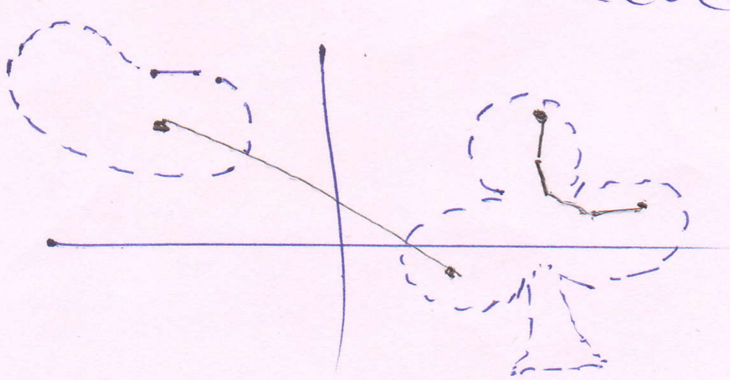


  — What about the following ?

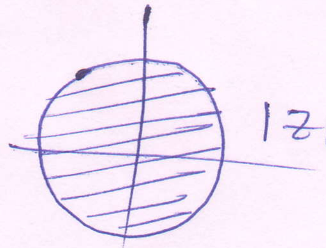


- An open set S is <u>connected</u> if any two of its points can be joined by a broken line of finitely many straight-line segments all of whose points belong to S.

  — Is this set connected ?



- Complement of a set S is the set of all points that do not belong to S
- A set S is closed if its complement is open,



$|z| \leq 1$ is closed,

- A boundary point of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S,
- An open connected set is called a domain,

Complex function

Let $S \subset \mathbb{C}$.

A function $f: S \to \mathbb{C}$ is a rule that assigns to every z in S a complex number w. Thus,

$$w = f(z).$$

(S is the domain of complex variable z).

e.g. $w = f(z) = 3z^2 - 7z + 9$,

(Set of all values of f is the

Let $W = u+iv$, $z = x+iy$ & $W = f(z)$, then

$$W = f(z) = u(x,y) + iv(x,y).$$

Limits and Continuity

We say $\lim\limits_{z \to z_0} f(z) = l$ if f is defined in a neighborhood of z_0 (except may be at z_0 itself) and if the values of f are "close" to l for all z "close" to z_0, Precisely,

given an $\varepsilon > 0$, $\exists\ \delta > 0 \ni \forall\ z \neq z_0$ in $|z - z_0| < \delta$, we have $|f(z) - l| < \varepsilon$.

- In case of limits of real variables, x can approach x_0 only from the left or from the right.
- In case of complex variables, z may approach z_0 from any direction.
- If a limit exists, it is unique.
- $f(z)$ is said to be continuous in a domain if it is continuous at each point of this domain, with continuity at $z = z$