MA 502 SEM. II (PART 2) – LECTURE 3

12/6/2020

ALREADY PROVED: Let f be analytic on an open disk G. If γ is any closed rectifiable curve in G, then $\int_\gamma f = 0$.

Question 1: For which regions G (other than the open disk G) does this result still remain valid?

Question 2: Fix a region G and an analytic function f on G. What should be the restrictions on a closed rectifiable curve γ so that $\int_\gamma f = 0$?

In this section, we answer question 2.

Lemma 5.3 Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$, let $F_m(z) := \int_\gamma \frac{\varphi(w)\,dw}{(w-z)^m}$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} - \{\gamma\}$ and $F_m'(z) = m F_{m+1}(z)$.

Proof: ① Claim: F_m is continuous for $m \geq 1$.

Note that in Thm. 5.2, we showed that the the index of γ w.r.t. a, where $a \in G$, given by

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-a},$$ is continuous at a.

To show $F_m(z) := \int_\gamma \frac{\varphi(w)\,dw}{(w-z)^m}$ is continuous for every $z \in G$, we follow a similar approach.

Note that since φ is continuous on a compact set $\{\gamma\}$, it is bounded. Moreover, using the expansion

$$x^m - y^m = (x-y)(x^{m-1} + x^{m-2}y + \dots + xy^{m-2} + y^{m-1}),$$

we see that

$$\frac{1}{(w-z)^m}-\frac{1}{(w-a)^m}=\left(\frac{1}{w-z}-\frac{1}{w-a}\right)\sum_{k=1}^{m}\frac{1}{(w-z)^{m-k}}\cdot\frac{1}{(w-a)^{k-1}}$$

$$=(z-a)\left\{\frac{1}{(w-z)^m(w-a)}+\frac{1}{(w-z)^{m-1}(w-a)^2}+\cdots+\frac{1}{(w-z)(w-a)^m}\right\}$$

Now we proceed as in the case of Thm. 5.2. — (*)
(Tutorial problem/ Homework problem).

Now fix $a\in G=\mathbb{C}-\{\gamma\}$ and let $z\in G \ni z\neq a$. Then

$$\frac{F_m(z)-F_m(a)}{z-a}=\frac{1}{z-a}\left\{\int_\gamma\frac{\varphi(w)\,dw}{(w-z)^m}-\int_\gamma\frac{\varphi(w)}{(w-a)^m}dw\right\}$$

$$=\int_\gamma\frac{\varphi(w)(w-a)^{-1}}{(w-z)^m}dw+\cdots+\int_\gamma\frac{\varphi(w)(w-a)^{-m}}{w-z}dw \quad \text{—} (**)$$

Now for $a\notin\{\gamma\}$, $\varphi(w)(w-a)^{-k}$ is continuous on $\{\gamma\}$ for each k. Hence by the first half of the proof (i.e; showing $\int_\gamma\frac{\varphi(w)}{(w-z)^m}dw$ is continuous for each $m\geq 1$ & for each $z\in G$), we see that each integral on the right-hand side of (**) is a continuous fn. of z for $z\in G$. Hence letting $z\to a$, we see that the limit exists whence we have

$$F_m'(a)=\int_\gamma\frac{\varphi(w)\,dw}{(w-a)^{m+1}}+\cdots+\int_\gamma\frac{\varphi(w)\,dw}{(w-a)^{m+1}}$$

$$=m\,F_{m+1}(a).$$

Thus we have shown that F_m is differentiable at a for each $m\geq 1$. Hence F_m is analytic ($\because F_m'(a)=m\,F_{m+1}(a)$)

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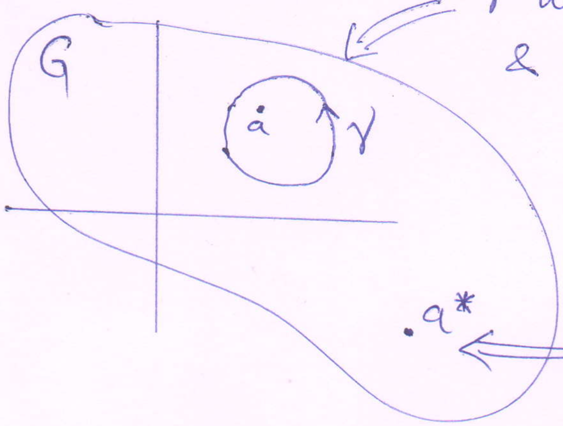
CAUCHY'S INTEGRAL FORMULA (1st version) (Thm. 5.4)

Let G be an open subset of the plane and $f: G \to \mathbb{C}$ be an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0 \;\forall w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$,

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z-a}.$$

Remarks: (1)



γ winds around a once, say. Then $n(\gamma; a) = 1$ & then we have $f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z-a}$, as seen previously.

$n(\gamma; a^*) = 0$. Hence

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a^*}\,dz = 0.$$

Proof: Let $\varphi: G \times G \to \mathbb{C}$ be defined by

$$\varphi(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w}, & z \neq w \\[2mm] f'(z), & z = w. \end{cases}$$

Since f is analytic on G (and hence continuous) & since $\varphi(z, z) = f'(z)$, we see that φ is continuous on $G \times G$.

Also, if we fix $z \in G$, then $z \to \varphi(z, w)$ is analytic. (Tutorial Set 9 problem).

Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is continuous and an integer-valued function of w, & $\{0\}$ is open, we have H to be open too.

Also $H \cup G = \mathbb{C}$ ($\because n(\gamma; w) = 0 \;\forall w \in \mathbb{C} \setminus G$).

(But $H \cap G$ may not be empty)

Define $g: \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int_\gamma \varphi(z,w)\,dw & , \text{ if } z \in G \\ \int_\gamma \frac{f(w)}{w-z}\,dw & , \text{ if } z \in H . \end{cases}$$

If $z \in H \cap G$, then

$$\underbrace{\int_\gamma \varphi(z,w)\,dw}_{\text{(exists since } \varphi \text{ is continuous in } w)} = \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw$$

$$= \int_\gamma \frac{f(w)\,dw}{w-z} - f(z)\int_\gamma \frac{dw}{w-z}$$

$$= \int_\gamma \frac{f(w)\,dw}{w-z} - f(z) \cdot 2\pi i\, n(\gamma; z)$$

$$= \int_\gamma \frac{f(w)\,dw}{w-z} .$$

Hence g is well-defined.

By Lemma 5.3, g is analytic in $\mathbb{C}$, i.e; an entire fn.

But by Thm. 5.2, H contains a neighborhood of ∞ in $\mathbb{C}_\infty$.

Now f analytic (hence continuous) on the compact set $\{\gamma\}$ implies f is bounded. Moreover,

$$\lim_{z\to\infty} \frac{1}{w-z} = 0 \quad \text{uniformly for } w \in \{\gamma\}. \text{ Hence,}$$

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \int_\gamma \frac{f(w)}{w-z}\,dw = 0 \quad \text{--- (1)}$$

(Note that $|g(z)| \le \int_\gamma \left|\frac{f(w)\,dw}{w-z}\right| \le M \int_\gamma \frac{|dw|}{|w-z|}$. Now let $z \to \infty$.)

So $\exists R > 0 \ni |g(z)| \le 1 \ \forall z \ni |z| \geq R$.
Since g being entire, obviously it's continuous on the compact set $\bar{B}(0;R)$, hence bounded.
Thus g is a bounded entire function, and hence a constant (Liouville's thm.)
But then from (1), $g \equiv 0$. Thus, if $a \in G \setminus \{\gamma\}$,

$$0 = \int_\gamma \frac{f(z) - f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)\, dz}{z-a} - f(a) \int_\gamma \frac{dz}{z-a}.$$

Hence, $\displaystyle\int_\gamma \frac{f(z)\, dz}{z-a} = n(\gamma; a) f(a)$.

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CAUCHY'S INTEGRAL FORMULA (2^{nd} version) (Thm. 5.5)

Let G be an open subset of $\mathbb{C}$ and let $f: G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G $\ni$ $n(\gamma_1; w) + \ldots + n(\gamma_m; w) = 0 \ \forall w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$,

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)\, dz}{z-a}.$$

Proof: Similar to that of the above theorem.
Just take $H = \{ w \in \mathbb{C} : n(\gamma_1; w) + \ldots + n(\gamma_m; w) = 0 \}$.

CAUCHY'S THEOREM (1^{st} version)

Let G be an open subset of $\mathbb{C}$ and $f: G \to \mathbb{C}$ be an analytic fn. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G $\ni$ $n(\gamma_1; w) + \ldots + n(\gamma_m; w) = 0 \ \forall w \in \mathbb{C} \setminus G$, then

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0.$$

Proof: Let ...