Derivative Let f be a function whose domain contains a neighborhood of a point z_0

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z},$$

provided the limit exists. Also written as, $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$, where $\Delta w = f(z+\Delta z) - f(z)$.

- f is then said to be differentiable at z_0.
- With $\Delta z = z - z_0$, we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

- Rules of differentiation same as in real calculus.

Eg. ① $f(z) = z^3$ is differentiable $\forall\, z \in \mathbb{C}$ since

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 - z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - \cancel{z^3}}{\Delta z}$$

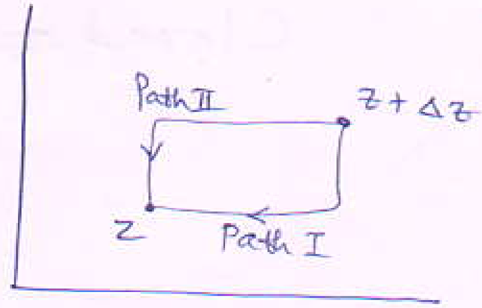
$$= 3z^2.$$

② $\bar{z}$ is not differentiable: Let $z = x+iy$ & Let $f(z) = \bar{z} = x - iy$. Suppose we write $\Delta z = \Delta x + i\Delta y$.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$,

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = 1$$



Analytic Functions

- A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

- f is said to be analytic at a point $z = z_0$ in D if f is analytic in a neighborhood of z_0.

  - Analyticity of f at z_0 means f has a derivative at every point in some neighborhood of z_0 (including z_0 itself).

  - Analytic or holomorphic mean the same

- Prove that $f(z) = |z|^2$ is nowhere analytic.

Sol^n: $\dfrac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \dfrac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$

$= z\dfrac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$ —— (*)

When $z = 0$, $\lim_{\Delta z \to 0} \dfrac{|z+\Delta z|^2 - |z|^2}{\Delta z}$ exists. Hence

Another way:

Let $\frac{\Delta w}{\Delta z} = z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$

- When Δz approaches the origin horizontally thro' $(\Delta x, 0)$ on the real axis:

$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z.$

$\Rightarrow \frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z. \Rightarrow \boxed{\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \bar{z} + z}$

- when Δz approaches the origin vertically thro' the points $(0, \Delta y)$ on the imaginary axis,

$\overline{\Delta z} = \overline{0 + i\Delta y} = -(0 + i\Delta y) = -\Delta z.$

$\Rightarrow \frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} - z$

$\Rightarrow \boxed{\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \bar{z} - z}$

Since the two limits must be same, we have

$\bar{z} + z = \bar{z} - z$

$\Rightarrow z = 0$, if $\frac{dw}{dz}$ is to exist.

CAUCHY - RIEMANN EQUATIONS, LAPLACE'S EQN.

- Is there a criterion to test if the function

$$w = f(z) = u(x,y) + i\,v(x,y)$$

is analytic in some domain ?

- Yes, f is analytic in a domain D "if and only if" (iff) (some additional hyp. reqd.) its real and imaginary parts satisfy the Cauchy-Riemann equations

$$\boxed{u_x = v_y \quad \text{and} \quad u_y = -v_x}$$

($u_x = \dfrac{\partial u}{\partial x}$, $u_y = \dfrac{\partial u}{\partial y}$, similarly for v).

- Prove that $f(z) = z^2$ is analytic in the whole complex plane.

<u>THM. 1</u> (Cauchy - Riemann equations)

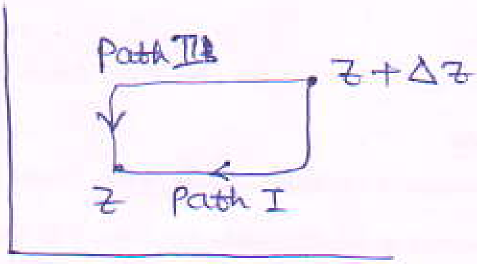
Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of $z = x+iy$ and differentiable at z itself. Then at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy - Riemann equations.

Hence $f(z) = u(x,y) + iv(x,y)$ analytic in a domain D

$\Downarrow$

Proof: By hypothesis, $f'(z)$ exists at z, that is,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists.}$$



- Let $z+\Delta z$ approach z first along path I. Let $\Delta z = \Delta x + i\Delta y$. So we first let $\Delta y \to 0$ and then $\Delta x \to 0$.

Now

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - (u(x,y) + iv(x,y))]}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\Rightarrow \boxed{f'(z) = u_x + i v_x} \quad \text{——} \quad (1)$$

Similarly along path II,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y) - v(x,y)}{i\Delta y}$$

$$\Rightarrow \boxed{f'(z) = -i u_y + v_y} \quad \text{——} \quad (2)$$