SUFFICIENT CONDITIONS FOR DIFFERENTI-ABILITY

Let the function $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ε- nbhd of a point $z_0 = x_0 + iy_0$, and suppose that the first-order partial derivatives of the functions u and v w.r.t. x and y exist everywhere in that nbhd. If those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x = v_y$ & $u_y = -v_x$, at (x_0, y_0), then $f'(z_0)$ exists.

- Showing $\bar{z}$ is not analytic using C-R eqns.

$\bar{z} = x - iy$. So $u(x,y) = x$ and $v(x,y) = -y$

$u_x = 1$ where as $v_y = -1$. So $u_x \neq v_y$.

$\Rightarrow \bar{z}$ is not analytic.

CONVERSE

THM 2 If two real-valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first ~~that~~ partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x,y) + i v(x,y)$ is analytic in D.

Proof:
- Quite involved; given in Appendix in Kreyszig's book
- Continuity of partial derivatives quite crucial in the proof.

Example: Let $f(z) = z^3 + z$.

$$= (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y).$$

So
$$\left.\begin{aligned} u(x,y) &= x^3 - 3xy^2 + x \\ v(x,y) &= 3x^2y - y^3 + y \end{aligned}\right\} \text{real-valued functions}$$

$u_x = 3x^2 - 3y^2 + 1$ $\qquad$ $u_y = -6xy$

$v_x = 6xy$ $\qquad$ $v_y = 3x^2 - 3y^2 + 1$

- If $f(z)$ is analytic in a domain D and $|f(z)| = k =$ constant in D, then $f(z) =$ constant in D.

<u>Proof</u>: $k^2 = |f(z)|^2 = u^2 + v^2$ (Note that $f = u + iv$)

$\Rightarrow \; uu_x + vv_x = 0 \quad \& \quad uu_y + vv_y = 0$

By C-R equations,

$$uu_x - vu_y = 0 \quad \text{and} \quad uu_y + vu_x = 0$$

$\Rightarrow \; u^2u_x - uvu_y = 0 \quad \text{and} \quad uvu_y + v^2u_x = 0$

Adding the two gives,

$$u_x(u^2+v^2) = 0 \quad \text{and similarly} \quad (u^2+v^2)u_y = 0.$$

If $k^2 = 0$, then $u^2+v^2 = 0 \Rightarrow u = v = 0$, so f is constant.

If $k \neq 0$, then $u_x = u_y = 0$ and by C-R,

$$v_x = v_y = 0 \text{ too.}$$

$\Rightarrow$ $u =$ constant & $v =$ constant

$\Rightarrow$ f is constant.

<u>Laplace's equation</u> & <u>Harmonic functions</u>

<u>Thm. 3</u> (Laplace's eqn.)

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace's eqn.

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

$$\& \quad \nabla^2 v = v_{xx} + v_{yy} = 0.$$

Proof: By C-R, $u_x = v_y$

$$\Rightarrow u_{xx} = \underset{\sim}{v_{yx}} \quad \text{— (a)}$$

(differentiate first w.r.t. y & then w.r.t. x)

Similarly $u_y = -v_x$

$$\Rightarrow u_{yy} = -v_{xy} \quad \text{— (b)}$$

Assuming for now, that the derivative of an analytic function is itself analytic, we find that u and v has continuous partial derivatives of all orders, in particular, the mixed second derivatives are equal: (by a thm. in advanced calculus) $v_{yx} = v_{xy}$ — (c)

From (a), (b) & (c), we have

$$u_{xx} + u_{yy} = 0$$

Similarly $v_{xx} + v_{yy} = 0$,

- Solutions of Laplace's eqn. having continuous second-order partial derivatives are called <u>harmonic functions</u> (very useful in <u>potential theory</u>)
- Hence the real & imaginary parts of an analytic fn. are harmonic functions.
- If two harmonic functions u and v satisfy the C-R eqns. in a domain D, they are the real & imaginary parts of an analytic function f in D

C-R equations in polar form

Let $z = r(\cos\theta + i\sin\theta)$

& $f(z) = u(r,\theta) + i\,v(r,\theta)$, then

$$\boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ \text{\&}\quad v_r &= -\frac{1}{r} u_\theta \end{aligned}} \qquad (r > 0)$$

~~CONVERSE~~

- Finding conjugate harmonic functions by the Cauchy–Riemann equations

- Determine whether $v = -e^{-x}\sin y$ is harmonic. If yes, find a corresponding analytic function $f(z) = u(x,y) + iv(x,y)$.

$v = -e^{-x}\sin y$

$\boxed{v_x = e^{-x}\sin y}$ $\quad v_{xx} = -e^{-x}\sin y$

$\boxed{v_y = -e^{-x}\cos y}$ $\quad v_{yy} = e^{-x}\sin y$

$\Rightarrow v_{xx} + v_{yy} = 0.$

$\Rightarrow v$ is harmonic.

To find harmonic conjugate of v:

If u is the one, then u & v satisfy C-R eqns

$\underline{u_x} = v_y = -\underline{e^{-x}\cos y}, \quad \underline{u_y} = -\underline{e^{-x}\sin y}$

$u = e^{-x}\cos y + h(y)$

$\Rightarrow u_y = -e^{-x}\sin y + h'(y) = -e^{-x}\sin y$

$\Rightarrow h'(y) = 0$

$\Rightarrow h(y) =$ constant, say c.

$\Rightarrow \boxed{u(x,y) = e^{-x}\cos y + c}$

$\Rightarrow f = u + iv$

$= (e^{-x}\cos y + c) + i(-e^{-x}\sin y)$

is analytic.