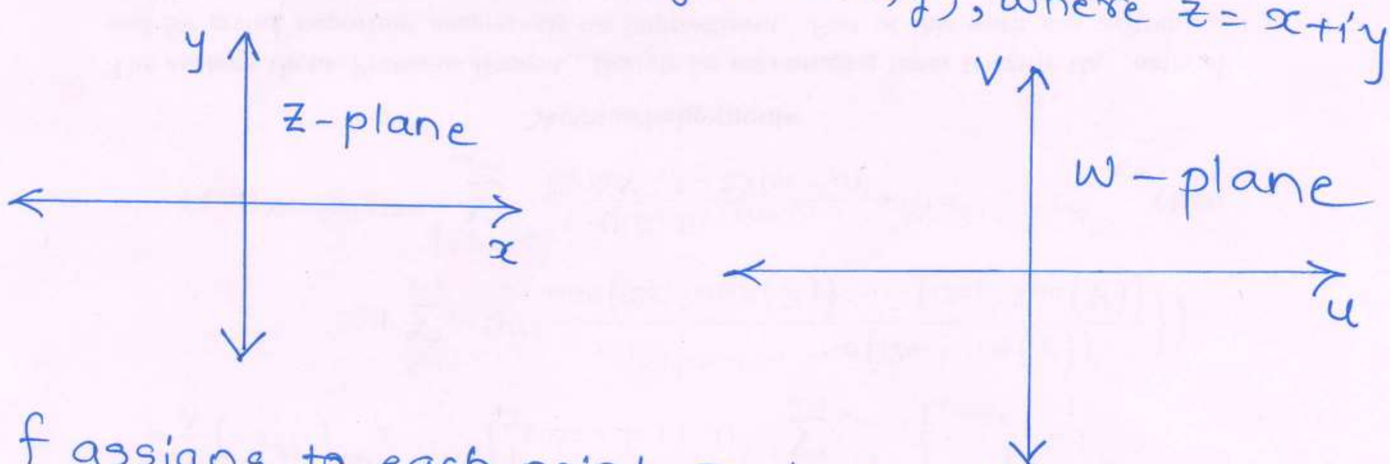
Finding conjugate harmonic functions by the Cauchy–Riemann equations

- Determine whether $v = -e^{-x}\sin y$ is harmonic. If yes, find a corresponding analytic function $f(z) = u(x,y) + iv(x,y)$.
- Determine a so that $u = \cosh ax \cos y$ is harmonic. Then find its harmonic conjugate.

Conformal Mapping

- Mapping: $w = f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$.



- f assigns to each point z in its domain of definition D the corresponding point $w = f(z)$ in the w-plane.

Example The mapping $w = z^2$.

- Let $z = r(\cos\theta + i\sin\theta)$ & $w = R(\cos\phi + i\sin\phi)$.

Then $w = R(\cos\phi + i\sin\phi) = r^2(\cos 2\theta + i\sin 2\theta)$

so that $R = r^2$ and $\phi = 2\theta$.

$u = \cosh ax \cos y$

$u_x = a \sinh ax \cos y$

$\boxed{u_{xx} = a^2 \cosh ax \cos y}$

$u_y = \cancel{asi} - \cancel{a}\cosh ax \sin y$

$\boxed{u_{yy} = -\cosh ax \cos y}$

$u = \cosh x \cos y$

$u_x = \boxed{\sinh x \cos y = v_y}$ — 1

$u_y = -\cosh x \sin y = -v_x$

$\Rightarrow \boxed{v_x = \cosh x \sin y}$ — 2

Integrate 2 w.r.t. x to get

$v = \sinh x \sin y + f(y)$

$\Rightarrow \boxed{v_y = \sinh x \cos y + \frac{df}{dy}}$ — 3

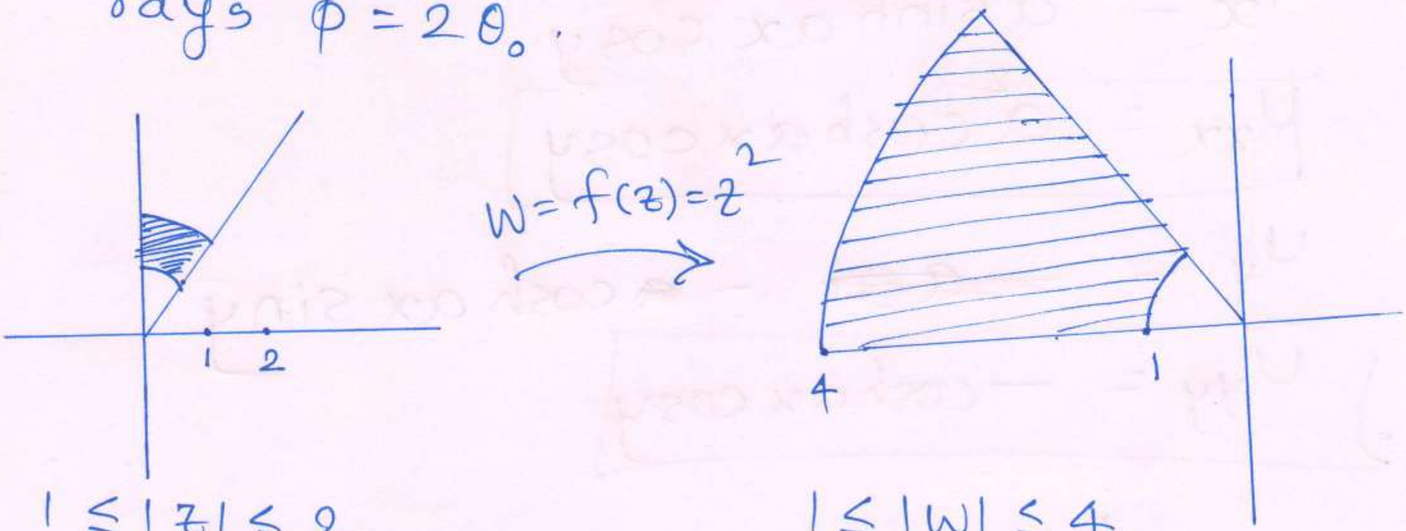
From 1 & 3, $\frac{df}{dy} = 0 \Rightarrow f = c$, constant

$v = \sinh x \sin y + c$

$f = \cosh x \cos y + i \sinh x \sin y + ic$

$= \cosh z + ic$

So $w = z^2$ maps circles $r = r_0$ onto circles $R = r_0^2$, and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$.



$1 \le |z| \le 2$ $\qquad\qquad$ $1 \le |w| \le 4$

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ $\qquad\qquad$ $\frac{2\pi}{3} \le \theta \le \pi$.

In Cartesian coordinates:

$$w = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) =: u + iv$$

So $u = x^2 - y^2$ and $v = 2xy$

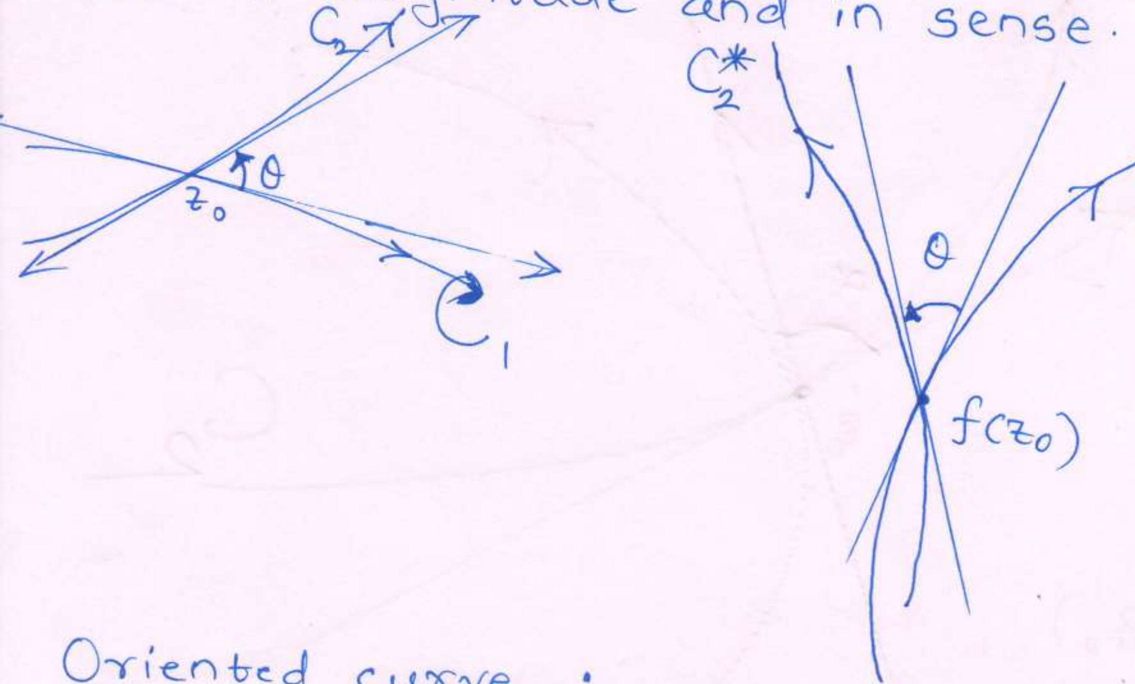
Let $x = c$, a constant. Then

$u = c^2 - y^2$ & $v = 2cy$

$$\Rightarrow v^2 = 4c^2(c^2 - u)$$

So $x = c$ gets mapped to these parabolas in w-plane. Similarly, for $y = d$, a constant, we get parabolas $v^2 = 4d^2(d^2 + u)$

- A <u>conformal</u> <u>mapping</u> is a mapping that preserves angles between any oriented curves, both in magnitude and in sense.



<u>Oriented curve</u>:

- A <u>parametric representation</u> for a curve C in xy-plane is $x = x(t)$, $y = y(t)$.

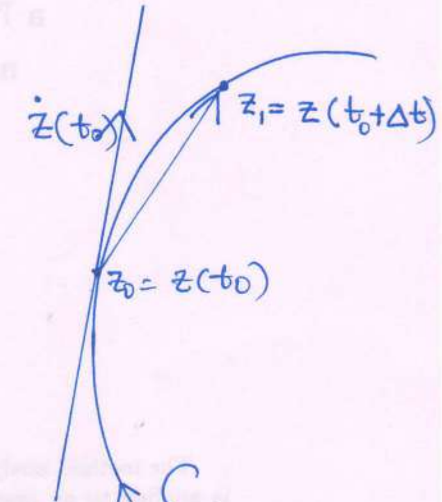
In the complex plane, $C: z(t) = x(t) + iy(t)$

- <u>Smooth curve C</u>: means $z(t)$ is differentiable and $\dot{z} = \frac{dz}{dt}$ is continuous & no-where zero.
- The sense of increasing values of the parameter t is called the <u>positive sense</u> on C. So $z(t)$ defines an orientation of C in this way.
- The angle of intersection θ between two curves C_1 & C_2 is defined as the angle between the oriented tan

Conformality of mapping by analytic functions

Thm. The mapping defined by an analytic function $f(z)$ is conformal except at critical points, that is, at points at which the derivative $f'(z)$ is zero.

Proof: — $\dot{z}(t) = \dfrac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$ as



it is $\displaystyle\lim_{\substack{z_1 \to z_0 \\ (\text{along } C)}} \frac{z_1 - z_0}{\Delta t}$.

The image C^* of C is $w = f(z(t))$.

By chain rule, $\dot{w} = f'(z(t))\,\dot{z}(t)$.

Thus, the tangent direction of C^* is given by $\boxed{\arg \dot{w} = \arg f' + \arg \dot{z}}$, where

$\arg \dot{z} =$ the tangent direction of C.

Thus, the mapping rotates ALL directions at a point z_0 in the domain of analyticity of f through the same angle $\arg f'(z_0)$, which exists as long as $f'(z_0) \neq 0$.

This implies conformality (because of rotation).

- The mapping $w = z + 1/z$.