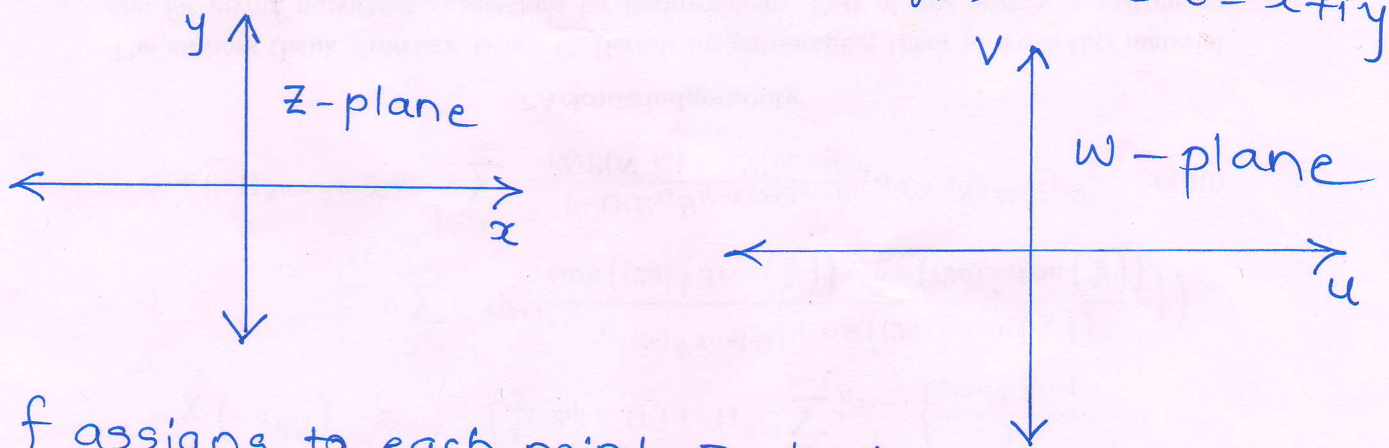
- Finding conjugate harmonic functions by the Cauchy–Riemann equations

- Determine whether $v = -e^{-x}\sin y$ is harmonic. If yes, find a corresponding analytic function $f(z) = u(x,y) + iv(x,y)$.

- Determine a so that $u = \cosh ax \cos y$ is harmonic. Then find its harmonic conjugate.

Conformal Mapping

- Mapping: $w = f(z) = u(x,y) + iv(x,y)$, where $z = x+iy$.



- f assigns to each point z in its domain of definition D the corresponding point $w = f(z)$ in the w-plane.

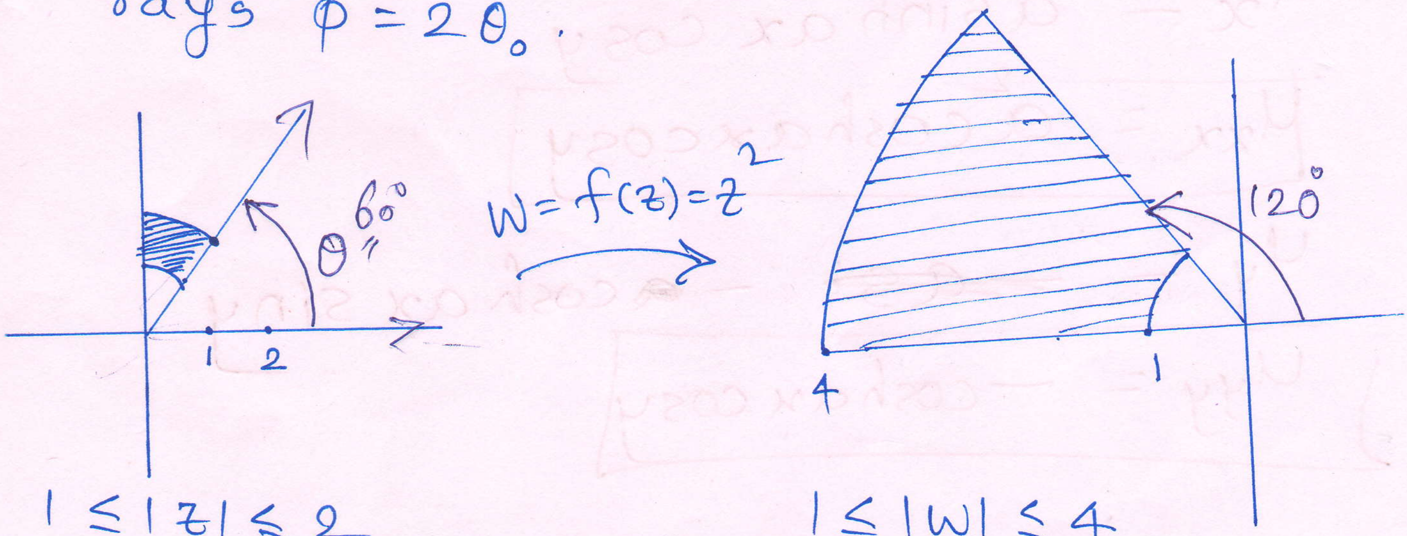
Example: The mapping $w = z^2$.

Let $z = r(\cos\theta + i\sin\theta)$ & $w = R(\cos\phi + i\sin\phi)$.

Then $w = R(\cos\phi + i\sin\phi) = r^2(\cos 2\theta + i\sin 2\theta)$

so that $R = r^2$ and $\phi = 2\theta$.

So $w = z^2$ maps circles $r = r_0$ onto circles $R = r_0^2$, and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$.



$1 \le |z| \le 2$ $\qquad\qquad$ $1 \le |w| \le 4$

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ $\qquad\qquad$ $\frac{2\pi}{3} \le \phi \le \pi$.

<u>In Cartesian coordinates</u>:

$$w = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) =: u + iv$$

So $u = x^2 - y^2$ and $v = 2xy$

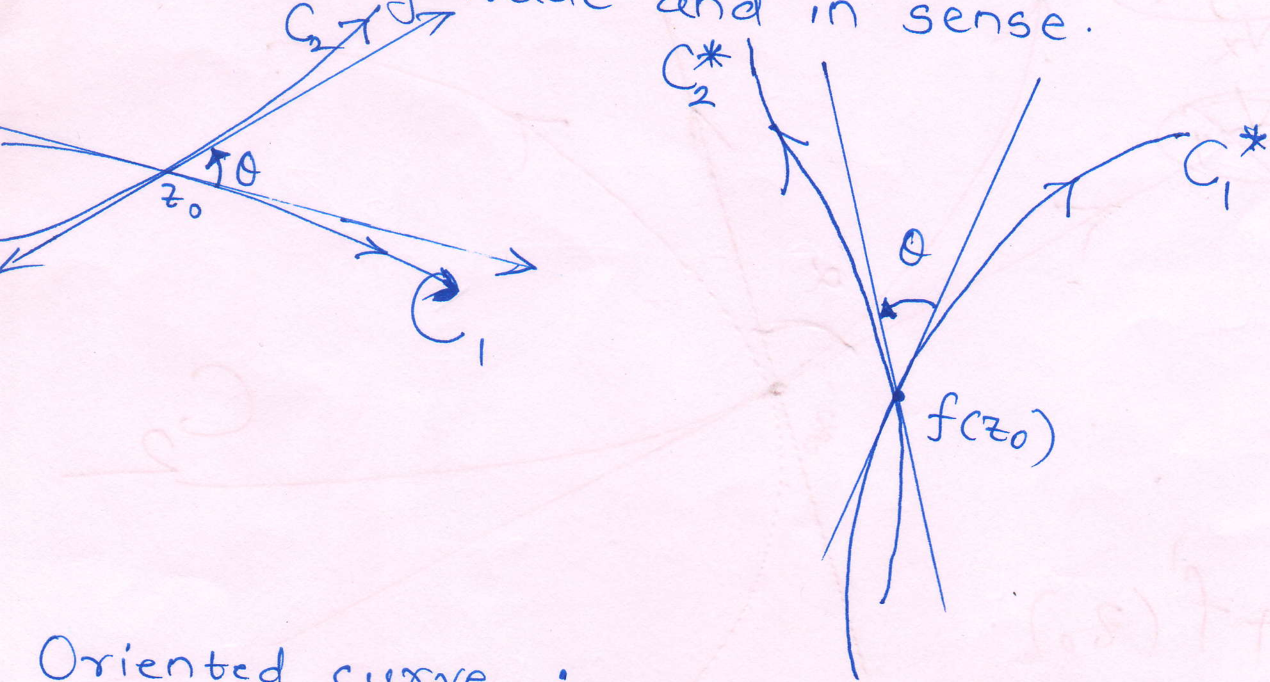
Let $x = c$, a constant. Then

$$u = c^2 - y^2 \ \& \ v = 2cy$$

$$\Rightarrow v^2 = 4c^2(c^2 - u)$$

So $x = c$ gets mapped to these parabolas in w-plane. Similarly, for $y = d$, a constant, we get parabolas $v^2 = 4d^2(d^2 + u)$

- A **conformal mapping** is a mapping that preserves angles between any oriented curves both in magnitude and in sense.



Oriented curve :

- A parametric representation for a curve C in xy-plane is $x = x(t)$, $y = y(t)$.

In the complex plane, $C: z(t) = x(t) + iy(t)$

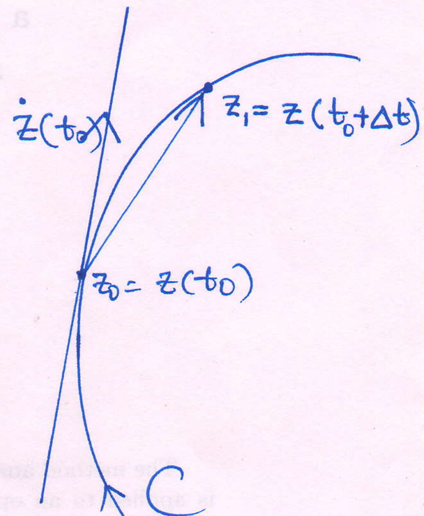
- Smooth curve C: means $z(t)$ is differentiable and $\dot{z} = \frac{dz}{dt}$ is continuous & no-where zero.
- The sense of increasing values of the parameter t is called the positive sense on C. So $z(t)$ defines an orientation of C in this way.
- The angle of intersection θ between two curves C_1 & C_2 is defined as the angle between the oriented tangents at the intersection point.

Conformality of mapping by analytic functions

Thm. The mapping defined by an analytic function $f(z)$ is conformal except at critical points, that is, at points at which the derivative $f'(z)$ is zero.

Proof:— $\dot{z}(t) = \dfrac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$ as

it is $\displaystyle\lim_{\substack{z_1 \to z_0 \\ (\text{along } c)}} \frac{z_1 - z_0}{\Delta t}$



The image C^* of C is $w = f(z(t))$.

By chain rule, $\dot{w} = f'(z(t))\,\dot{z}(t)$

Thus, the tangent direction of C^* is given by $\boxed{\arg \dot{w} = \arg f' + \arg \dot{z}}$, where

$\arg \dot{z} =$ the tangent direction of C.

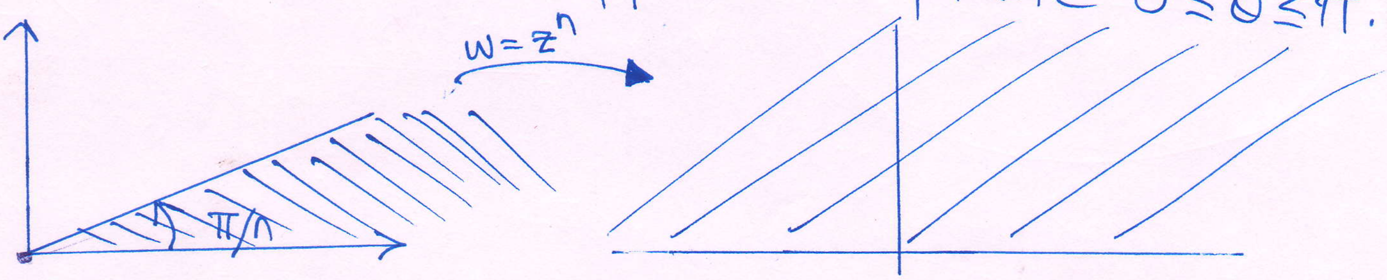
Thus, the mapping rotates <u>ALL</u> directions at a point z_0 in the domain of analyticity of f through the <u>same</u> angle $\arg f'(z_0)$, which exists as long as $f'(z_0) \neq 0$.

This implies conformality (because of rotation).

• The mapping $w = z + 1/z$.

- $w = z^2$, even though analytic everywhere, is not conformal at $z=0$ as $w' = 2z = 0$ at $z=0$.

- In general, ~~for~~ $w = z^n$ will map the sector $0 \le \theta \le \pi/n$ to the upper half plane $0 \le \theta \le \pi$.



- Joukowski's transformation $w = z + 1/z$.

Let $z = r(\cos\theta + i\sin\theta)$. Then

$$w = u+iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

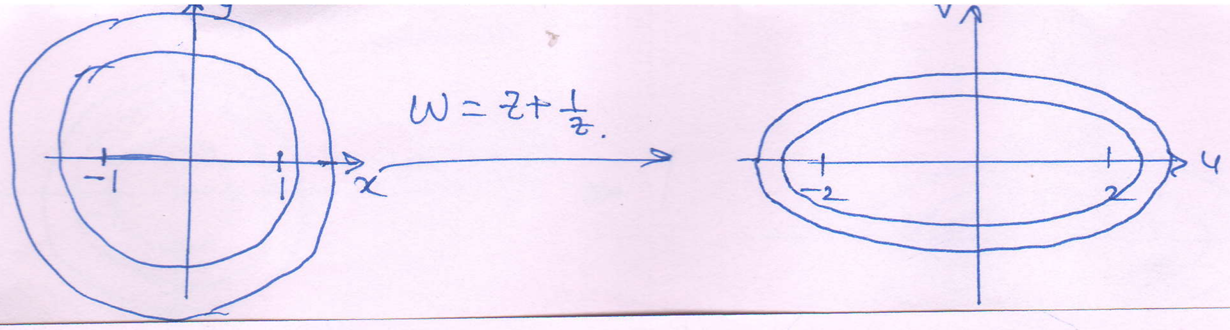
So $u = a\cos\theta$, $v = b\sin\theta$, where

$a = r + 1/r$, $b = r - 1/r$.

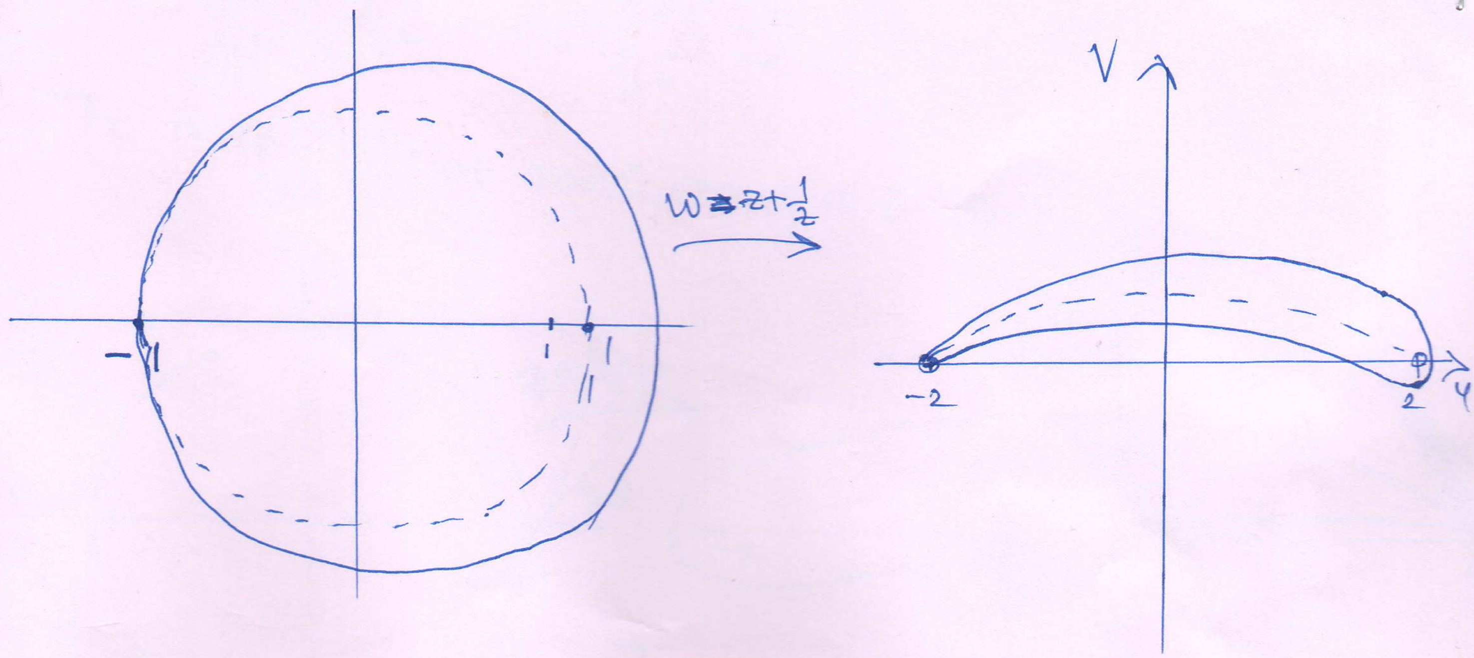
Thus $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$.

Hence the circles $|z| = r \ne 1$ get mapped to ellipses $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$.

- when $r=1$, $w = 2\cos\theta$. Since $-1 \le \cos\theta \le 1$, $-2 \le u \le 2$. Hence the unit circle gets mapped to the segment $-2 \le u \le 2$.



y
x
-1
1
w = z + 1/z.
v
u
-2
2



-1
1
w = z + 1/z
v
u
-2
2