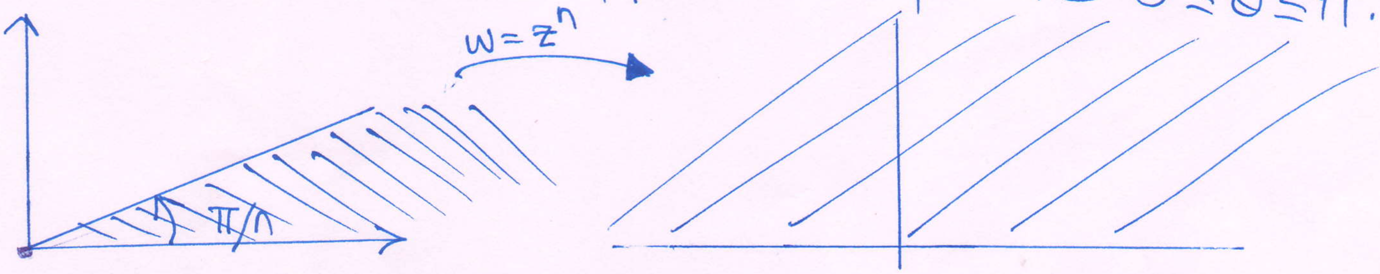
- $w = z^2$, even though analytic everywhere, is not conformal at $z=0$ as $w' = 2z = 0$ at $z = 0$.

- In general, ~~for~~ $w = z^n$ will map the sector $0 \le \theta \le \pi/n$ to the upper half plane $0 \le \theta \le \pi$.



- <u>Joukowski's transformation</u> $w = z + 1/z$.

Let $z = r(\cos\theta + i\sin\theta)$. Then

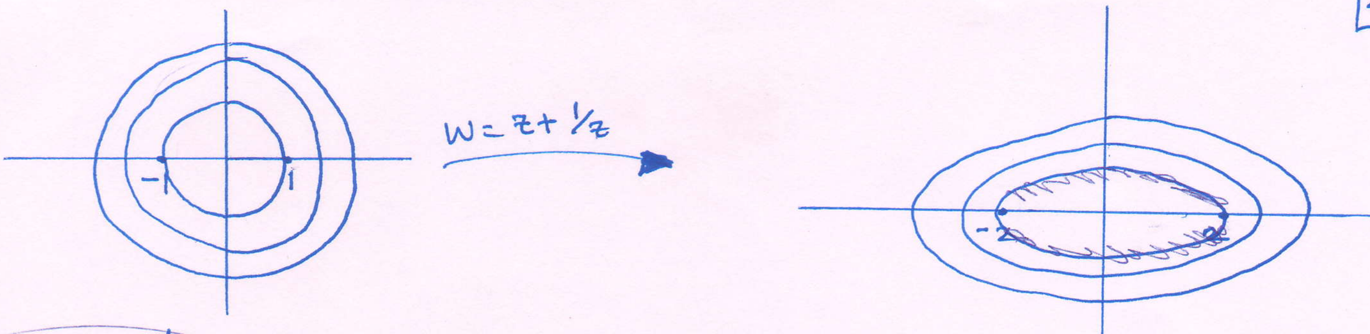
$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

So $u = a\cos\theta$, $v = b\sin\theta$, where
$a = r + 1/r$, $b = r - 1/r$.

Thus $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$.

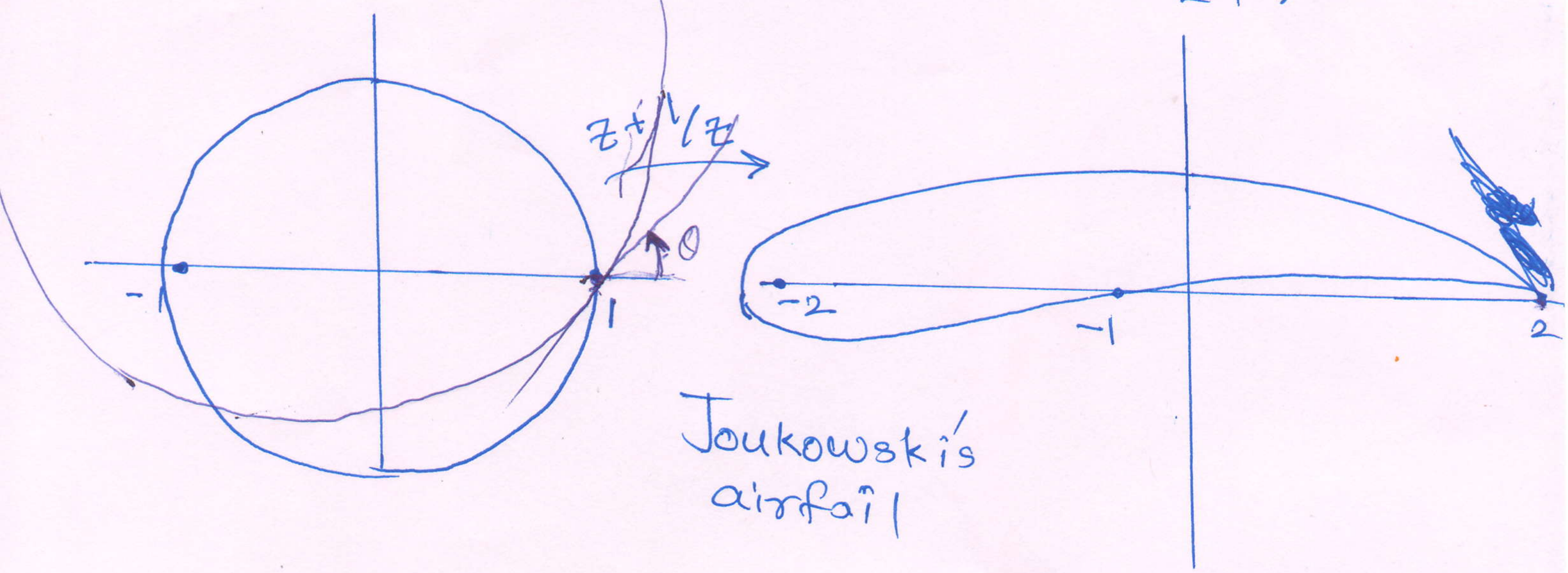
Hence the circles $|z| = r \neq 1$ get mapped to ellipses $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$.

- when $r = 1$, $w = 2\cos\theta$. Since $-1 \le \cos\theta \le 1$, $-2 \le u \le 2$. Hence the unit circle gets mapped to the segment $-2 \le u \le 2$.



$$w' = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2}$$

The map is not conformal at $z = \pm 1$.



Joukowski's airfoil

Section 13.5 – Exponential function

Defn. $e^z = e^x(\cos y + i \sin y)$

Motivation: (i) e^z should reduce to its real counterpart, when $z = x$ is real,

(ii) e^z should be an entire function, that is, analytic for all z.

(iii) Need $(e^z)' = e^z$.

(i) — Easy to prove

(ii) $u = e^x \cos y$, $v = e^x \sin y$

$u_x = e^x \cos y = v_y$ & $u_y = -e^x \sin y = -v_x$

- $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$e^{z_1} e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1) \cdot e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1+x_2}\left(\cos(y_1+y_2) + i\sin(y_1+y_2)\right)$$
$$= e^{z_1+z_2}$$

- So if $z_1 = x$ & $z_2 = iy$,

$$e^{z_1+z_2} = e^{x} \cdot e^{iy}$$

Let $z = x+iy$. Then

$$e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}.$$

Now if $x = 0$, we have Euler's formula

$$\boxed{e^{iy} = \cos y + i\sin y.} \quad \text{---} (*)$$

- Now the polar form of a complex number can be written in the form

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

- Also from (*), (a) $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi)$
$$= 1$$

(b) $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$,

(c) Similarly, $e^{\pi i/2} = i$, $e^{-i\pi/2} = -i$, $e^{-\pi i} = -1$.

- Also, $|e^{iy}| = |\cos y + i\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$.

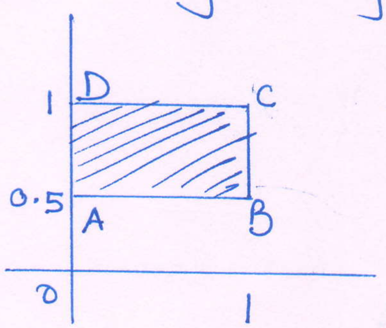
Thus, $|e^{z}| = |e^{x+iy}| = |e^{x} \cdot e^{iy}| = e^{x} \cdot 1 = e^{x}$,

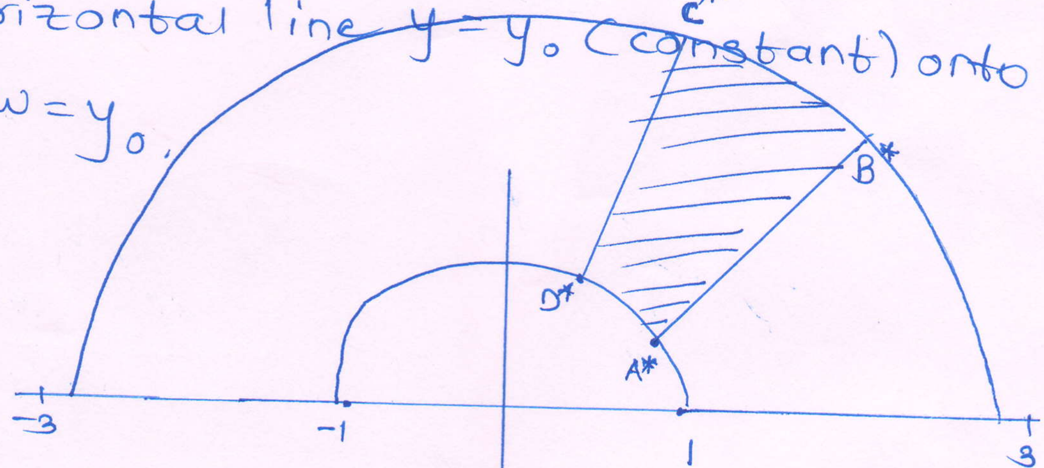
Hence ... z

- $|e^z| = e^x \neq 0$. Hence $e^z \neq 0$.

$\boxed{\text{Remark}}$: Thus e^z is an entire function that never vanishes. This is in contrast to non-constant polynomials which, though entire, always vanish at some point.

- e^z maps a vertical line $x = x_0$ (constant) onto the circle $|w| = e^{x_0}$
- e^z maps a horizontal line $y = y_0$ (constant) onto the ray $\arg w = y_0$.

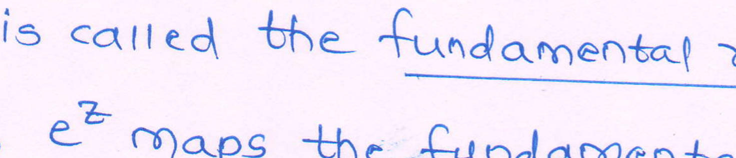




Periodicity of e^z with period $2\pi i$

$$e^{z+2\pi i} = e^z \quad \forall z.$$

- All values assumed by $w = e^z$ are assumed if z lies in the horizontal strip of width $2\pi \ni -\pi < y \leq \pi$



This is called the fundamental region of e^z.

- Thus, e^z maps the fundamental region bijectively onto the complex plane (excludin