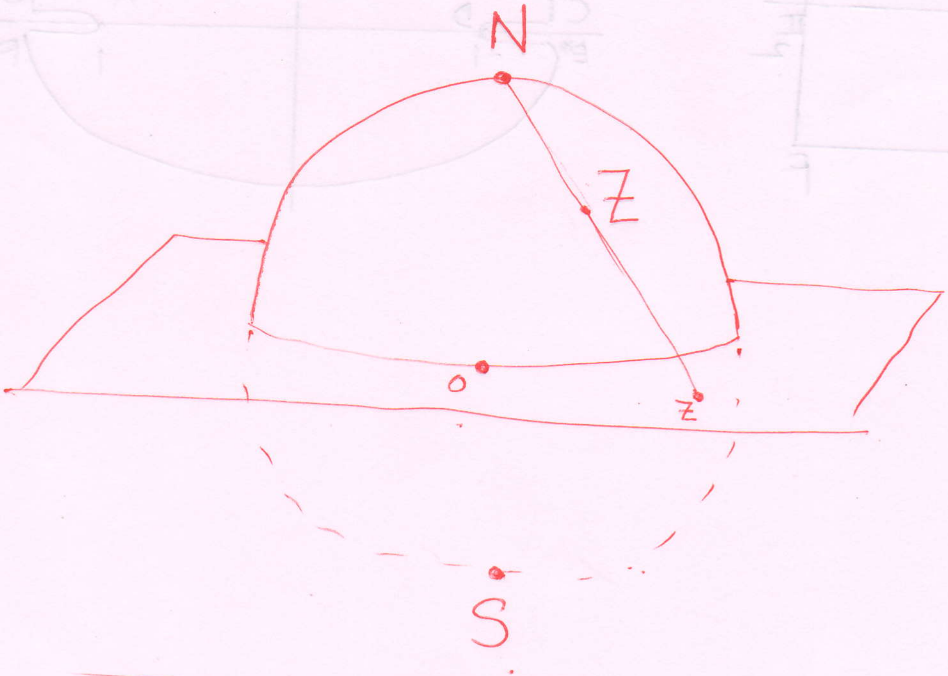
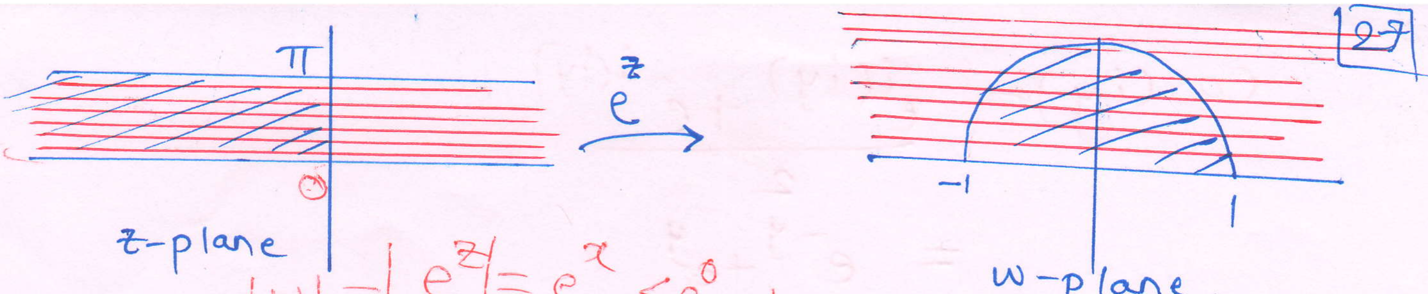
Stereographic projection & compactification





$|w| = |e^z| = e^x \le e^0 = 1$

Chapter 12.7 - Trigonometric functions & Hyperbolic functions

- would like the real trigonometric functions to be extended to complex ones.
- Note that $e^{ix} = \cos x + i \sin x$

$$e^{-ix} = \cos x - i \sin x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \& \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- Motivated by this, we define
  - $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$
  - $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
- We then define $\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{\cos z}{\sin z}$

$$\sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z \ (\csc z) = \frac{1}{\sin z}.$$

- $\cos z$ & $\sin z$ are entire functions of z since e^z is.
- $\tan z$, $\sec z$, $\cot z$, $\operatorname{cosec} z$ are NOT enti[...]

- $e^{iz} = \cos z + i\sin z$ holds for $z \in \mathbb{C}$ too.

Real and imaginary parts, absolute value & periodicity

- Show that if $z = x+iy$, then
  1. $\cos z = \cos x \cosh y - i \sin x \sinh y$
  2. $\sin z = \sin x \cosh y + i \cos x \sinh y$
  3. $|\cos z|^2 = \cos^2 x + \sinh^2 y$
  4. $|\sin z|^2 = \sin^2 x + \sinh^2 y$

- Note that for x real, $|\cos x| \le 1$ & $|\sin x| \le 1$. But for z complex, $|\cos z|$ & $|\sin z|$ tend to infinity as $y \to \infty$. So $\cos z$ & $\sin z$ are unbounded.

- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.
- $\cos^2 z + \sin^2 z = 1$
- $\frac{d}{dz}(\cos z) = -\sin z$, $\frac{d}{dz}(\sin z) = \cos z$.

Hyperbolic functions

- $\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$
- $\dfrac{d}{dz}\cosh z = \sinh z$, $\dfrac{d}{dz}\sinh z = \cosh z$
- $\tanh z = \dfrac{\sinh z}{\cosh z}$, $\coth z = \dfrac{\cosh z}{\sinh z}$

$\operatorname{sech} z = \dfrac{1}{\cosh z}$, $\operatorname{cosech} z$ ($\operatorname{csch} z$) $= \dfrac{1}{\sinh z}$.

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2}$$

$$= \frac{e^{-y} + e^{y}}{2}$$

$$= \cosh y$$

$$|\cos z|^2 = \cos^2 x \, \underline{\cosh^2 y} + \underline{\sin^2 x} \sinh^2 y$$

$$= \cos^2 x (1 + \sinh^2 y) + (1 - \cos^2 x) \sinh^2 y$$

$$= \cos^2 x + \sinh^2 y$$

$$\sinh y = \frac{e^{y} - e^{-y}}{2} \xrightarrow[y \to \infty]{} \infty$$

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \cdot i$$

$$= i \sin z$$

- $\cosh(iz) = \cos z$, $\sinh(iz) = i\sin z$
- $\cos(iz) = \cosh z$, $\sin(iz) = i\sinh z$.

Conformal mapping of $\sin z$, $\cos z$, $\sinh z$, $\cosh z$

$$W = u+iv = \sin z = \sin(x+iy)$$
$$= \sin x \cosh y + i\cos x \sinh y$$

$\Rightarrow$ $\boxed{u = \sin x \cosh y \quad , \quad v = \cos x \sinh y}$

$\Rightarrow$ ① If x = constant (i.e. a line parallel to y-axis) (imaginary),

then so are $\sin x$ & $\cos x$.

Then $\dfrac{u}{\sin x} = \cosh y$, $\dfrac{v}{\cos x} = \sinh y$.

Since $\cosh^2 y - \sinh^2 y = 1$, we have

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \qquad \text{(Hyperbolas)}$$

② If y = constant (a line parallel to real axis)

So are $\cosh y$ and $\sinh y$.

$$\frac{u}{\cosh y} = \sin x \quad \& \quad \frac{v}{\sinh y} = \cos x$$

Since $\sin^2 x + \cos^2 x = 1$, we have

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \qquad \text{(Ellipses)}$$

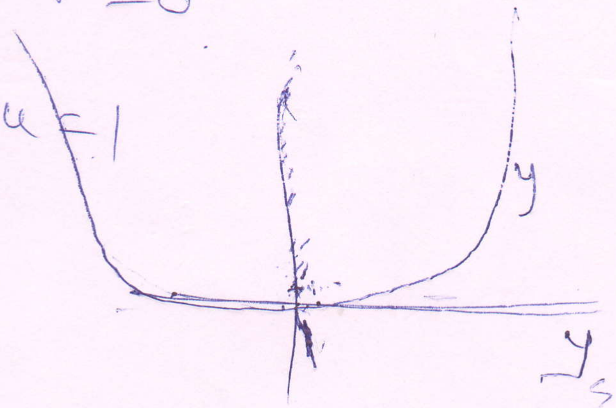
- The ellipses and hyperbolas intersect each other at right angles.

- Exceptions are when $x = \pm \frac{\pi}{2}$. They get folded onto $u \geq 1$ and $u \leq -1$ respectively.

$u = \sin x \cosh y \qquad v = \cos x \sinh y$

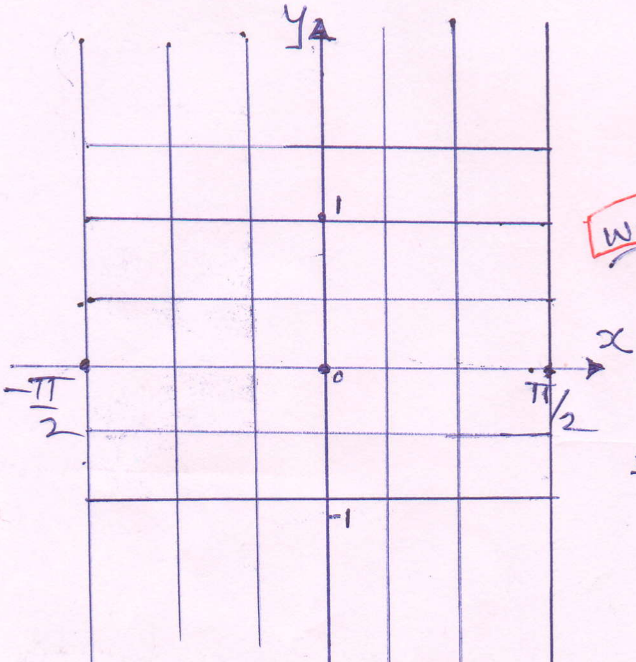
when $x = \frac{\pi}{2}$, $\boxed{u = \cancel{\sin x}\ \cosh y}$, $v = 0$

when $y = 0$, $u = 1$

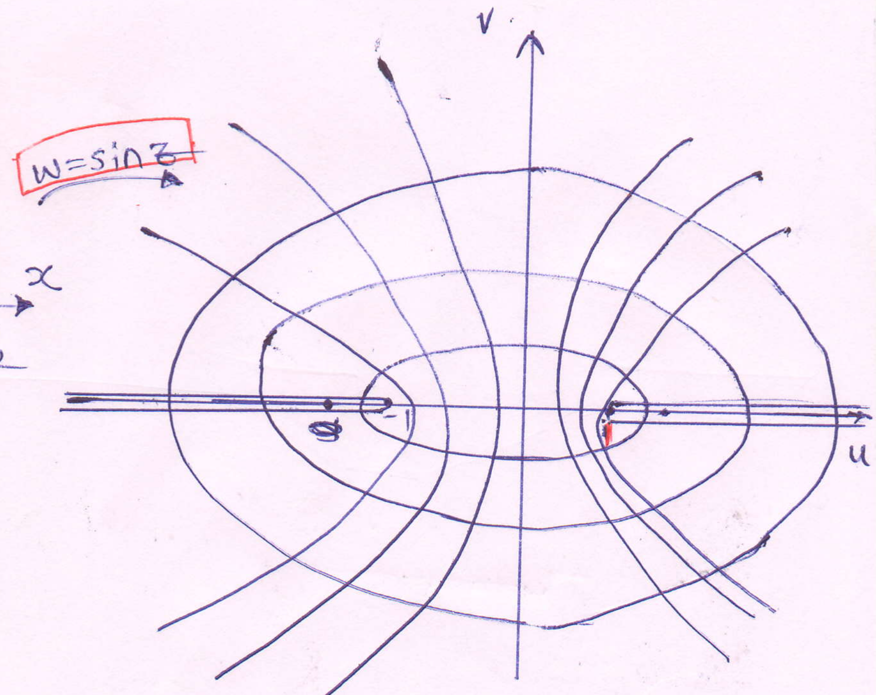


$\{y, -10, 10\}$

$\{y, -0.05, 0.05$



(z-plane)



w-plane