MA 502 – (SEM. II – PART 2) – LECTURE 7 — 23/6/2020

LAST TIME: If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$.

Cor. 5.13 If γ is a closed rectifiable curve in G s.t. $\gamma \sim 0$, then $n(\gamma; w) = 0 \ \forall w \in \mathbb{C} \setminus G$.

Proof: Take γ_1 to be a constant curve in G and $\gamma_0 = \gamma$. Then $\gamma \sim 0$. Now $\int_\gamma f = 0$. Finally, take $f(z) = \frac{1}{z-w}$. This function is analytic in $\mathbb{C} \setminus G$. Hence

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w} = 0 \Rightarrow n(\gamma; w) = 0 \ \forall w \in \mathbb{C} \setminus G.$$ ∎

Converse NOT TRUE: $\exists$ a closed rectifiable curve in $G \ni n(\gamma; w) = 0 \ \forall w \in \mathbb{C} \setminus G$ but $\gamma \not\sim 0$. (HW problem)

- If G is an open set and γ_0 and γ are closed rectifiable curves in G, then $n(\gamma_0; a) = n(\gamma_1; a)$ for each $a \in \mathbb{C} \setminus G$. Provided $\gamma_0 \sim \gamma_1 (G)$.

Example: Let $\gamma_0(t) = e^{2\pi i t}$ and $\gamma_1(t) = e^{-2\pi i t}$ for $0 \le t \le 1$. Then $n(\gamma_0; 0) = 1$ & $n(\gamma_1; 0) = -1 \Rightarrow \gamma_0 \not\sim \gamma_1 (G)$, where $G = \mathbb{C} \setminus \{0\}$.

Defn. If $\gamma_0, \gamma_1 : [0,1] \to G$ are 2 rectifiable curves in $G \ni \gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$, then γ_0 and γ_1 are fixed-end-point homotopic (FEP homotopic) if there is a continuous map $\Gamma : I^2 \to G \ni$

$$\Gamma(s,0) = \gamma_0(s), \quad \Gamma(s,1) = \gamma_1(s)$$
$$\Gamma(0,t) = a, \quad \Gamma(1,t) = b$$

for $0 \le s, t \le 1$.

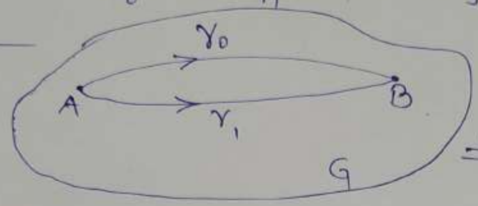
- FEP homotopy is an equivalence relation on curves from one given point to another (PROVE!)

INDEPENDENCE OF PATH THEOREM

If γ_0 and γ_1 are two rectifiable curves in G from a to b and γ_0 and γ_1 are FEP homotopic, then

$$\int_{\gamma_0} f = \int_{\gamma_1} f \text{ for any analytic function } f \text{ in } G.$$

Proof:



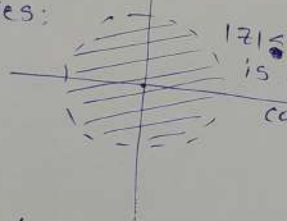
$\gamma_0 - \gamma_1$ is a closed rectifiable curve in G.

$\Rightarrow \int_{\gamma_0} f - \int_{\gamma_1} f = 0$ ∎

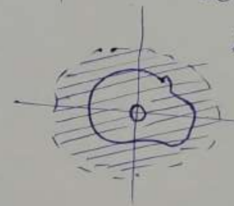
- We now characterize those regions G for which the integral of an analytic function around a closed curve is always zero.

Defn. An open set G is simply connected if G is connected and every closed curve in G is homotopic to zero.

Examples:



$|z| \le 1$ is simply connected



$\{z : |z| < 1\} \setminus \{0\}$ is not simply connected.

CAUCHY'S THEOREM (Fourth version) Thm. 5.14

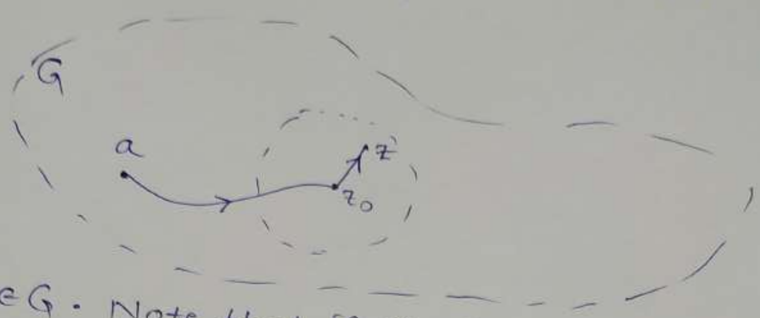
If G is simply connected, then $\int_\gamma f = 0$ for every closed rectifiable curve and every analytic function f.

Example: Complement of the spiral $r = \theta$ is simply connected.

- A region G is simply connected iff its complement in the extended plane, $\mathbb{C}_\infty \setminus G$, is connected in $\mathbb{C}_\infty$.
- The domain of the principal branch of the logarithm is simply connected.

Cor. 5.15 If G is simply connected and $f: G \to \mathbb{C}$ is analytic in G, then f has a primitive in G,

Proof:



Fix $a \in G$. Note that if γ_1, γ_2 are any two rectifiable curves in G from 'a' to a point 'z' in G. By the 4th version of Cauchy's theorem,

$$0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f \Rightarrow \int_{\gamma_1} f = \int_{\gamma_2} f .$$

Hence this gives a well-defined function $F: G \to \mathbb{C}$ by setting $F(z) = \int_{\gamma} f$ where γ is any rectifiable curve from a to z.

Claim: F is a primitive of f.

If $z_0 \in G$ & $r > 0 \ni B(z_0; r) \subset G$, then let γ be a path from a to z_0. For z in $B(z_0; r)$, let $\gamma_z = \gamma + [z_0, z]$.

Hence
$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{(z - z_0)} \int_{[z_0, z]} f .$$

Now the argument is similar to that in the proof of Morera's theorem which gives $F'(z_0) = f(z_0)$.

$\boxtimes$

Cor. 5.16 Let G be simply connected and let $f: G \to \mathbb{C}$ be an analytic such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g: G \to \mathbb{C}$ such that $f(z) = \exp(g(z))$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Proof: Since f never vanishes, $\frac{f'}{f}$ is analytic on G. Hen by previous corollary, it has a primitive g_1. If $h(z) = e^{g_1(z)}$ then h is a non-vanishing analytic function. So f/h is analytic and

$$(f/h)' = \frac{hf' - fh'}{h^2}$$

But $h' = \frac{d}{dz}(e^{g_1(z)}) = e^{g_1(z)} \cdot g_1'(z) = hg' = \frac{hf'}{f}$.

$\Rightarrow fh' - hf' = 0 \Rightarrow (f/h)' = 0$

$\Rightarrow f/h$ is constant c $\forall z \in G$

$\Rightarrow f(z) = ch(z) = c e^{g_1(z)} = e^{g_1(z) + c'}$ for some c'

Now let $g(z) = g_1(z) + c' + 2\pi i k$ for an appropriate k, with $g(z_0) = w_0$, we arrive at the desired result.

Cor. 5.17 $\gamma \sim 0 \Rightarrow \gamma \approx 0$.