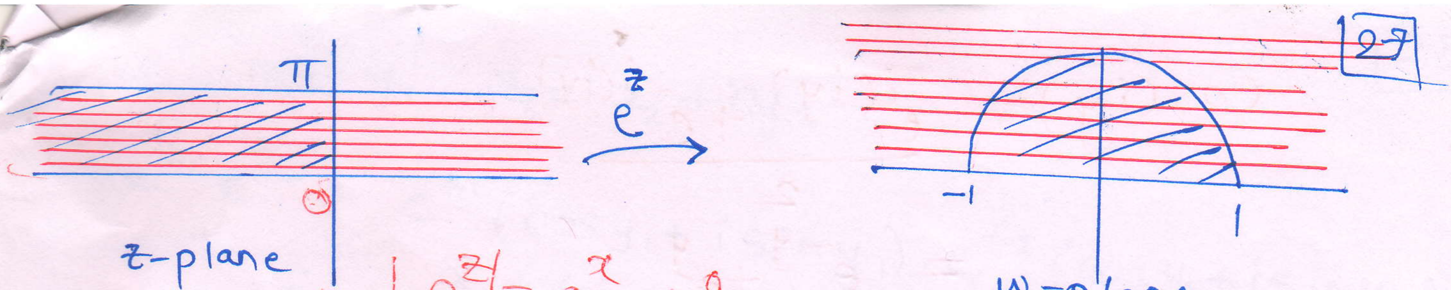


$|w| = |e^z| = e^x \le e^0 = 1$

Chapter 12.7 – Trigonometric functions & Hyperbolic functions

- would like the real trigonometric functions to be extended to complex ones.
- Note that $e^{ix} = \cos x + i\sin x$

$$e^{-ix} = \cos x - i\sin x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \& \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- Motivated by this, we define
  - $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$
  - $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
- We then define $\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{\cos z}{\sin z}$

$$\sec z = \frac{1}{\cos z}, \quad \begin{matrix}\text{cosec } z \\ (\csc z)\end{matrix} = \frac{1}{\sin z}.$$

- $\cos z$ & $\sin z$ are entire functions of z since e^z is.
- ∴ $\tan z, \sec z, \cot z, \text{cosec } z$ are NOT entire functions of z.

- $e^{iz} = \cos z + i\sin z$ holds for $z \in \mathbb{C}$ too.

Real and imaginary parts, absolute value & periodicity

- Show that if $z = x+iy$, then
  1. $\cos z = \cos x \cosh y - i\sin x \sinh y$
  2. $\sin z = \sin x \cosh y + i\cos x \sinh y$
  3. $|\cos z|^2 = \cos^2 x + \sinh^2 y$
  4. $|\sin z|^2 = \sin^2 x + \sinh^2 y$

- Note that for x real, $|\cos x| \leq 1$ & $|\sin x| \leq 1$. But for z complex, $|\cos z|$ & $|\sin z|$ tend to infinity as $y \to \infty$. So $\cos z$ & $\sin z$ are unbounded.

- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.
- $\cos^2 z + \sin^2 z = 1$
- $\frac{d}{dz}(\cos z) = -\sin z$, $\frac{d}{dz}(\sin z) = \cos z$.

Hyperbolic functions

- $\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$
- $\dfrac{d}{dz}\cosh z = \sinh z$, $\dfrac{d}{dz}\sinh z = \cosh z$
- $\tanh z = \dfrac{\sinh z}{\cosh z}$, $\coth z = \dfrac{\cosh z}{\sinh z}$

  $\text{sech}\, z = \dfrac{1}{\cosh z}$, $\text{cosech}\, z$ $(\text{csch}\, z)$ $= \dfrac{1}{\sinh z}$.

- $\cosh(iz) = \cos z$, $\sinh(iz) = i\sin z$
- $\cos(iz) = \cosh z$, $\sin(iz) = i\sinh z$.

Conformal mapping of $\sin z$, $\cos z$, $\sinh z$, $\cosh z$

$$W = u + iv = \sin z = \sin(x+iy)$$
$$= \sin x \cosh y + i\cos x \sinh y$$

$\Rightarrow$ $\boxed{u = \sin x \cosh y \quad , \quad v = \cos x \sinh y}$

$\Rightarrow$ ① If x = constant (i.e. a line parallel to y-axis) (imaginary),

then so are $\sin x$ & $\cos x$.

Then $\frac{u}{\sin x} = \cosh y$, $\frac{v}{\cos x} = \sinh y$.

Since $\cosh^2 y - \sinh^2 y = 1$, we have

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \quad \text{(Hyperbolas)}$$

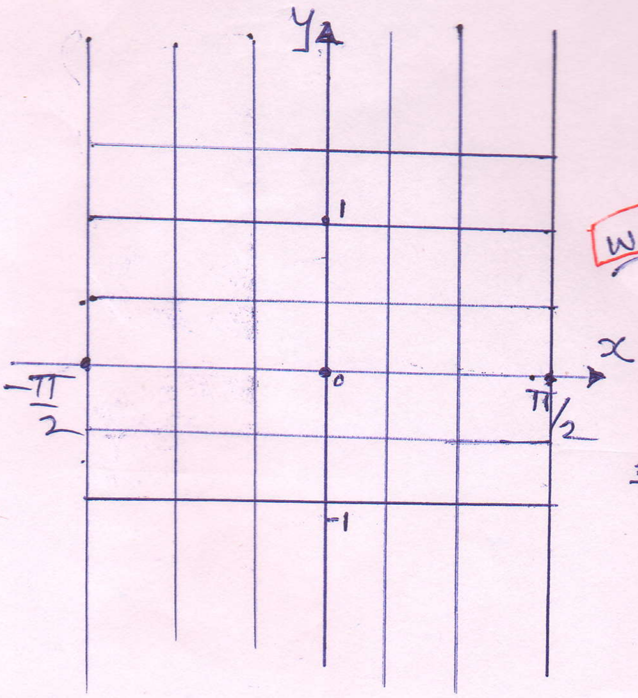
② If y = constant (a line parallel to real axis),
So are $\cosh y$ and $\sinh y$.

$$\frac{u}{\cosh y} = \sin x \quad \& \quad \frac{v}{\sinh y} = \cos x$$

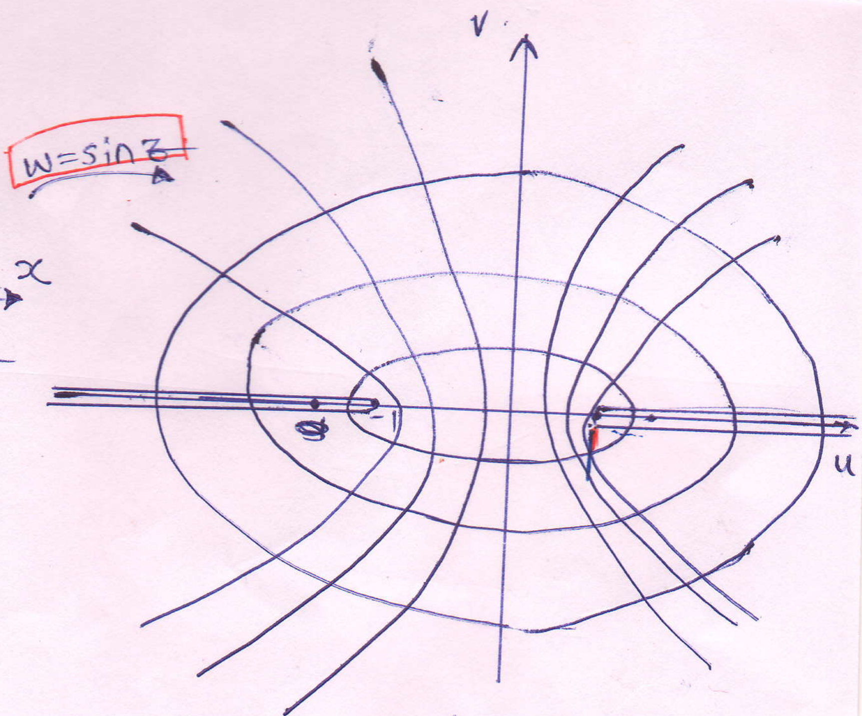
Since $\sin^2 x + \cos^2 x = 1$, we have

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \quad \text{(Ellipses)}$$

- The ellipses and hyperbolas intersect each other at right angles.

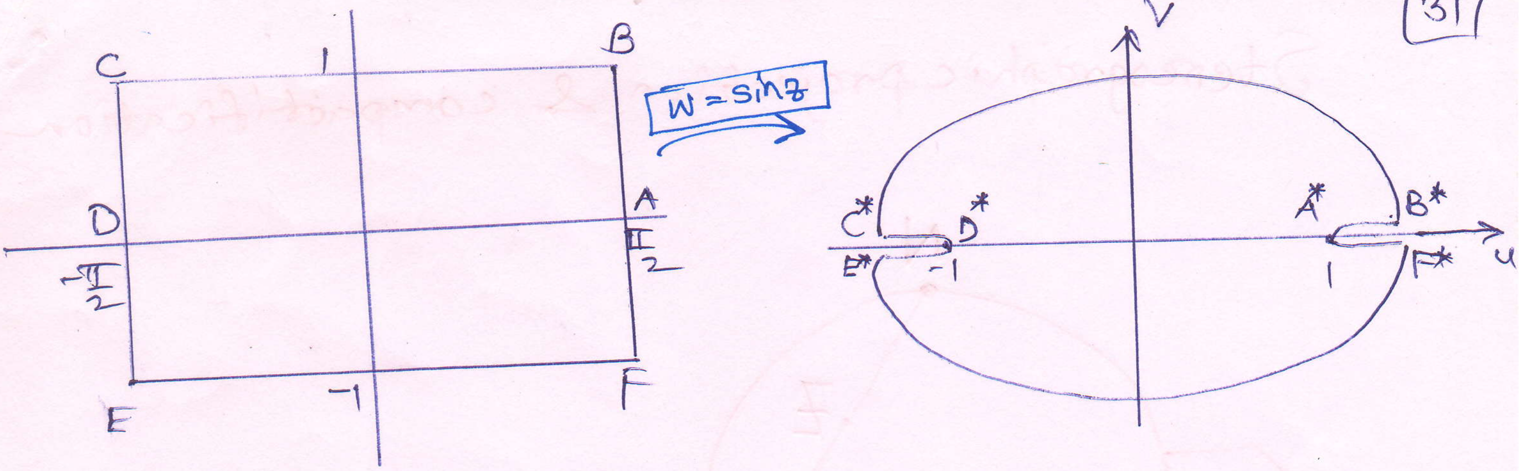


(z-plane)



w-plane

We took $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ because



- Upper & lower sides of the rectangle mapped onto semi-ellipses.
- At the vertical line segment: $x = \frac{\pi}{2}$, $-1 \le \boxed{y \le 1}$

  Note that $u = \sin x \cosh y$, $v = \cos x \sinh y$

  $\Rightarrow$ $u = \cosh y$, $v = 0$. We know that $u \ge 1$.

  Since $y \le 1$, $u = \cosh y \le \cosh 1$. Hence $\boxed{1 \le u \le \cosh 1}$.
- Similarly, $x = -\frac{\pi}{2}$ gets mapped onto $\boxed{-\cosh 1 \le u \le -1}$.
- The mapping is not conformal at $u = \pm 1$. (WHY)?

$\boxed{\text{Discuss also } w = \cos z,\ w = \sinh z \text{ and } w = \cosh z}$.

$$\cos z = \sin\left(z + \frac{\pi}{2}\right) = \sin z \circ \left\{z \to z + \frac{\pi}{2}\right\}$$

$$\sinh z = -i \sin(iz) \qquad \boxed{i = e^{i\pi/2}}$$

$$\cosh z = \cos(iz) \qquad iz = z e^{i\pi/2}$$

$$= r e^{i(\theta + \pi/2)}$$