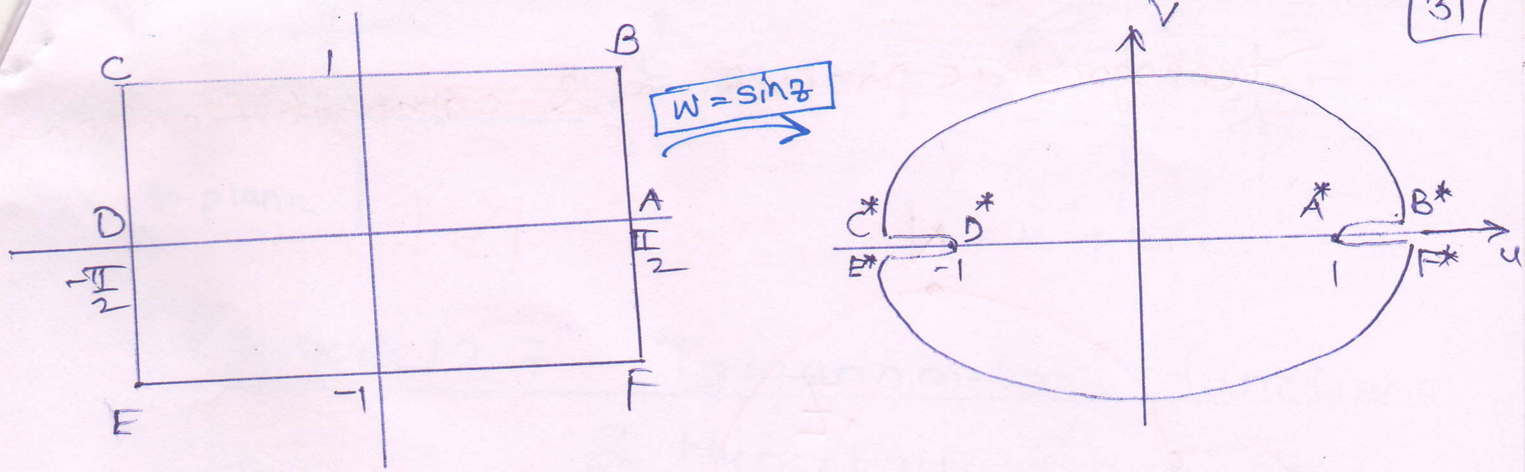


- Upper & lower sides of the rectangle mapped onto Semi-ellipses.
- At the vertical line segment: $x = \frac{\pi}{2}$, $\boxed{-1 \le y \le 1}$
  Note that $u = \sin x \cosh y$, $v = \cos x \sinh y$
  $\Rightarrow$ $u = \cosh y$, $v = 0$. We know that $u \ge 1$.
  Since $y \le 1$, $u = \cosh y \le \cosh 1$. Hence $\boxed{1 \le u \le \cosh 1}$.
- Similarly, $x = -\frac{\pi}{2}$ gets mapped onto $\boxed{-\cosh 1 \le u \le -1}$.
- The mapping is not conformal at $u = \pm 1$. (WHY) ?

$\boxed{\text{Discuss also } w = \cos z,\ w = \sinh z \text{ and } w = \cosh z}$.

$$\cos z = \sin\left(z + \frac{\pi}{2}\right) = \sin z \circ \left\{z \to z + \frac{\pi}{2}\right\}$$

$$\sinh z = -i \sin(iz) \qquad \boxed{i = e^{i\pi/2}}$$

$$\cosh z = \cos(iz) \qquad iz = z e^{i\pi/2}$$

$$= r e^{i(\theta + \pi/2)}$$

## Sect. 13.8 Logarithm, general power

- Natural logarithm of $z = x+iy$, denoted by $\ln z$ (or $\log z$) is defined as the inverse of the exponential function, i.e., $w = \ln z$ is defined for $z \neq 0$ by $e^w = z$.

- $w = u+iv$, $z = re^{i\theta}$. Then

$$e^w = e^{u+iv} = e^u \cdot e^{iv} = re^{i\theta}.$$

$$\Rightarrow e^u = r, \quad v = \theta \quad (!)$$

Now $e^u = r \Rightarrow u = \ln r = \ln|z|$.
(real natural logarithm)

$$\Longrightarrow \ln z = \ln r + i\theta \qquad (r = |z| > 0, \ \theta = \arg z)$$

- But argument of $z$ is determined ONLY upto integer multiples of $2\pi$.

$\Longrightarrow$ The complex natural logarithm $\ln z$ ($z \neq 0$) is many-valued function, in fact, infinitely many-valued.

- $\operatorname{Ln} z$ : $\ln z$ corresponding to principal value $\operatorname{Arg} z$.
(principal value of $\ln z$).

$$\Rightarrow \operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z \qquad (z \neq 0).$$

Single-valued function

Also $\ln z = \operatorname{Ln} z \pm 2n\pi i \qquad (n \in \mathbb{N})$.

- $z$ positive real, say $z = x > 0$ : $\operatorname{Ln} z = \log x$ (or $\ln x$)
- $z$ negative real: $\operatorname{Ln} z = \ln|z| + \pi i$.

Properties of $\ln z$ :

① $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

② $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$.

Remark: Let $z_1 = z_2 = e^{\pi i} = -1$.

$\operatorname{Ln} z_1 = \operatorname{Ln} z_2 = \pi i$

$\Rightarrow \operatorname{Ln} z_1 + \operatorname{Ln} z_2 = 2\pi i$

So ① will hold provided we write
$\ln(z_1 z_2) = \ln 1 = 2\pi i$.

But it won't hold if we take $\operatorname{Ln}(z_1 z_2) = \operatorname{Ln} 1 = 0$.

* $\ln z = \ln r + i\theta$ (where $z = re^{i\theta}$)

$\Rightarrow e^{\ln z} = e^{\ln r + i\theta} = e^{\ln r} \cdot e^{i\theta} = re^{i\theta} = z$

$\Rightarrow \boxed{e^{\ln z} = z}$

* However, $\arg(e^z) = y \pm 2n\pi$, $n \in \mathbb{N}$
(multi-valued)

$\Rightarrow \boxed{\ln(e^z) = z \pm 2n\pi i}$

Since $\ln(e^z) = \ln|e^z| + i \arg(e^z)$

& $\ln|e^z| = \ln(e^x)$