Sect. 13.8 Logarithm, general power

- Natural logarithm of $z = x+iy$, denoted by $\ln z$ (or $\log z$) is defined as the inverse of the exponential function, i.e., $w = \ln z$ is defined for $z \neq 0$ by $e^w = z$.

- $w = u+iv$, $z = re^{i\theta}$. Then

$$e^w = e^{u+iv} = e^u \cdot e^{iv} = re^{i\theta}.$$

$$\Rightarrow e^u = r, \quad v = \theta \quad (!)$$

Now $e^u = r \Rightarrow u = \ln r = \ln|z|$.
(real natural logarithm)

$$\Longrightarrow \ln z = \ln r + i\theta \qquad (r = |z| > 0, \ \theta = \arg z)$$

- But argument of z is determined ONLY upto integer multiples of 2π.

$\Rightarrow$ The complex natural logarithm $\ln z$ $(z \neq 0)$ is many-valued function, in fact, infinitely many-valued.

- $\mathrm{Ln}\, z$: $\ln z$ corresponding to principal value $\mathrm{Arg}(z)$.
(principal value of $\ln z$).

$$\Rightarrow \mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z \qquad (z \neq 0).$$

Single-valued function

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Also $\ln z = \text{Ln}\, z \pm 2n\pi i \quad (n \in \mathbb{N})$.

- z positive real, say $z = x > 0$: $\text{Ln}\, z = \log x$ (or $\ln x$)
- z negative real: $\text{Ln}\, z = \ln|z| + \pi i$.

Properties of $\ln z$:

① $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

② $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$.

Remark: Let $z_1 = z_2 = e^{\pi i} = -1$.

$\text{Ln}\, z_1 = \text{Ln}\, z_2 = \pi i$

$\Rightarrow \text{Ln}\, z_1 + \text{Ln}\, z_2 = 2\pi i$

So ① will hold provided we write
$\ln(z_1 z_2) = \ln 1 = 2\pi i$.

But it won't hold if we take $\text{Ln}(z_1 z_2) = \text{Ln}\, 1 = 0$.

* $\ln z = \ln r + i(\theta + 2n\pi)$ (where $z = re^{i\theta}$)

$\Rightarrow e^{\ln z} = e^{\ln r + i(\theta + 2n\pi)} = e^{\ln r} \cdot e^{i\theta} = re^{i\theta} = z$

$\Rightarrow \boxed{e^{\ln z} = z}$

* However, $\arg(e^z) = y \pm 2n\pi$, $n \in \mathbb{N}$ (multi-valued)

$\Rightarrow \boxed{\ln(e^z) = z \pm 2n\pi i}$

Since $\ln(e^z) = \ln|e^z| + i\arg(e^z)$

& $\ln|e^z| = \ln(e^x) = x \ln e = x$.

$= x + i(y \pm 2n\pi) = z \pm 2n\pi i$

- $\ln z = \operatorname{Ln} z \pm 2n\pi i$

(For a fixed nonnegative integer n, this ~~rep~~ defines a function.)

- ~~These~~ The above functions are analytic in $\mathbb{C}\setminus\{z\in\mathbb{R} : \cancel{\operatorname{Re}(z)}\ z \le 0\}$.

<u>Proof</u>: Note that $\ln z$ is not defined at $z=0$. Also, $\ln z$ is not continuous on the negative real axis because its imaginary part has a jump discontinuity of 2π there. Hence it is not analytic on the negative real axis.

For $\mathbb{C}\setminus\{z\in\mathbb{R} : \cancel{\operatorname{Re}(z)}\ z \le 0\}$, we show that $\ln z$ is analytic by proving $\frac{d}{dz}\ln z = \frac{1}{z}$, that is, $\ln z$ is differentiable at all points in $\mathbb{C}\setminus\{z : \cancel{\operatorname{Re}(z)}\ z \le 0\}$.

Note that
$$\ln z = \ln|z| + i\theta$$
$$= \frac{1}{2}\ln(x^2+y^2) + i\left(\arctan\frac{y}{x} + c\right)$$
(c is a multiple of 2π).

$$\Rightarrow u = \frac{1}{2}\ln(x^2+y^2), \quad v = \arctan\left(\frac{y}{x}\right) + c$$

$$u_x = \frac{x}{x^2+y^2}, \quad v_y = \frac{1}{1+y^2/x^2}\cdot\frac{1}{x}$$

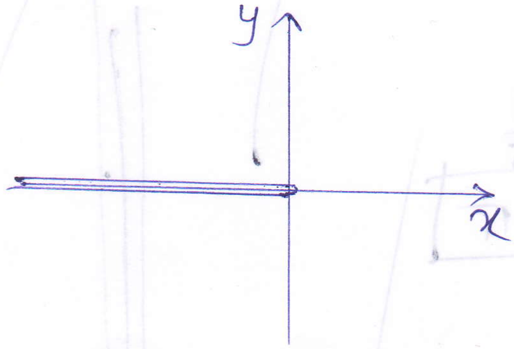
$$u_y = \frac{y}{x^2+y^2}, \quad -v_x = -\frac{1}{1+(y/x)^2}\cdot\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow (\ln z)' = u_x + i v_x = \frac{x}{x^2+y^2} + i\,\frac{1}{1+(y/x)^2}\cdot\left(\frac{-y}{x^2}\right) = \frac{x-iy}{x^2+y^2} = 1/z.$$

- For every $n \in \mathbb{Z}$, the function
  $\ln(z) = \operatorname{Ln}(z) + 2\pi n i$ is called a <u>branch</u> of the logarithm.

- The negative real axis is known as a <u>branch cut.</u>

- The branch for $n=0$ is called the <u>principal branch</u> of $\ln(z)$.



General Powers

Def. Let $z = x+iy$.

$$z^c :\doteq e^{c \ln z} \quad (c \in \mathbb{C},\ z \neq 0)$$

- $\ln z$ infinitely many-valued implies so is z^c.
- $z^c = e^{c \operatorname{Ln} z}$ is the principal value of z^c.
- For $c = 1, 2, 3, \ldots$; z^c is single-valued & identical with n^{th} power of z. Similarly for $c = -1, -2, -3, \ldots$
- If $c = \frac{1}{n}$, $n = 1, 2, 3, \ldots$, then

$$z^c = \sqrt[n]{z} = e^{\frac{1}{n} \ln z}$$ (Unique exponent up to multiples of $\frac{2\pi i}{n}$)

$\frac{2\pi i m}{n}$

$\Rightarrow$ n distinct values of the n^{th} root.

- If $c = p/q$, again finitely many distinct values
- If c is irrational or complex, z^c is infinitely many-valued