- $\ln z = \mathrm{Ln}\, z \pm 2n\pi i$

(For a fixed nonnegative integer n, this defines a function.)

- The above functions are analytic in $\mathbb{C} \setminus \{ z \in \mathbb{R} : z \leq 0 \}$.

Proof: Note that $\ln z$ is not defined at $z=0$. Also, $\ln z$ is not continuous on the negative real axis because its imaginary part has a jump discontinuity of 2π there. Hence it is not analytic on the negative real axis.

For $\mathbb{C} \setminus \{ z \in \mathbb{R} : z \leq 0 \}$, we show that $\ln z$ is analytic by proving $\frac{d}{dz} \ln z = \frac{1}{z}$, that is, $\ln z$ is differentiable at all points in $\mathbb{C} \setminus \{ z : z \leq 0 \}$.

Note that
$$\ln z = \ln|z| + i\theta$$
$$= \frac{1}{2}\ln(x^2+y^2) + i\left(\arctan\frac{y}{x} + c\right)$$
(c is a multiple of 2π).

$$\Rightarrow u = \frac{1}{2}\ln(x^2+y^2), \quad v = \arctan\left(\frac{y}{x}\right) + c$$

$$u_x = \frac{x}{x^2+y^2}, \quad v_y = \frac{1}{1+y^2/x^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2+y^2}, \quad -v_x = -\frac{1}{1+(y/x)^2} \cdot \left(\frac{-y}{x^2}\right)$$

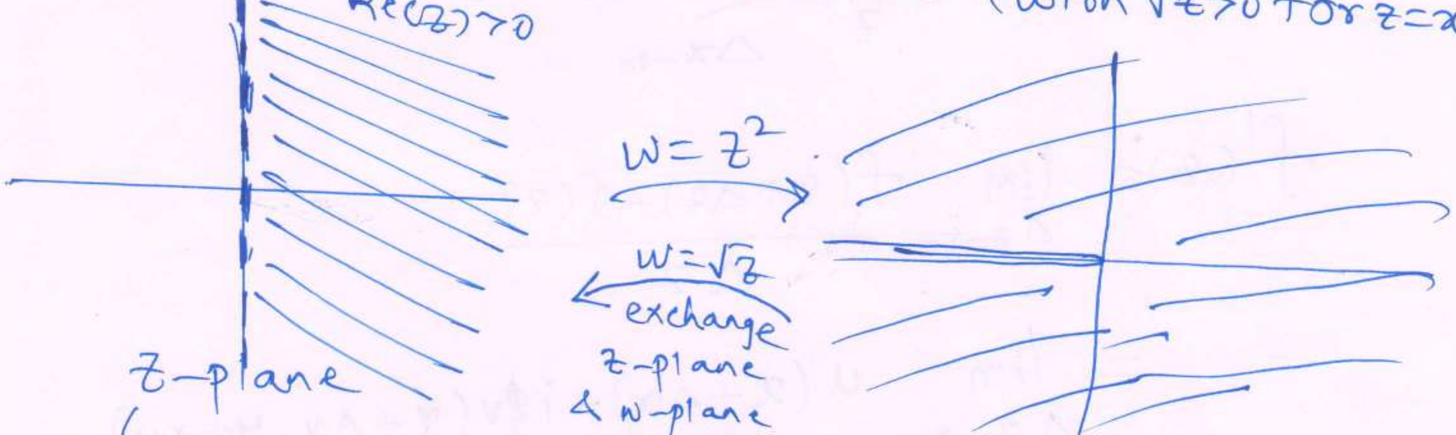
$$\Rightarrow (\ln z)' = u_x + i v_x = \frac{x}{x^2+y^2} + i\,\frac{1}{1+(y/x)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{x-iy}{x^2+y^2}$$

Conformal mapping by $\ln z$

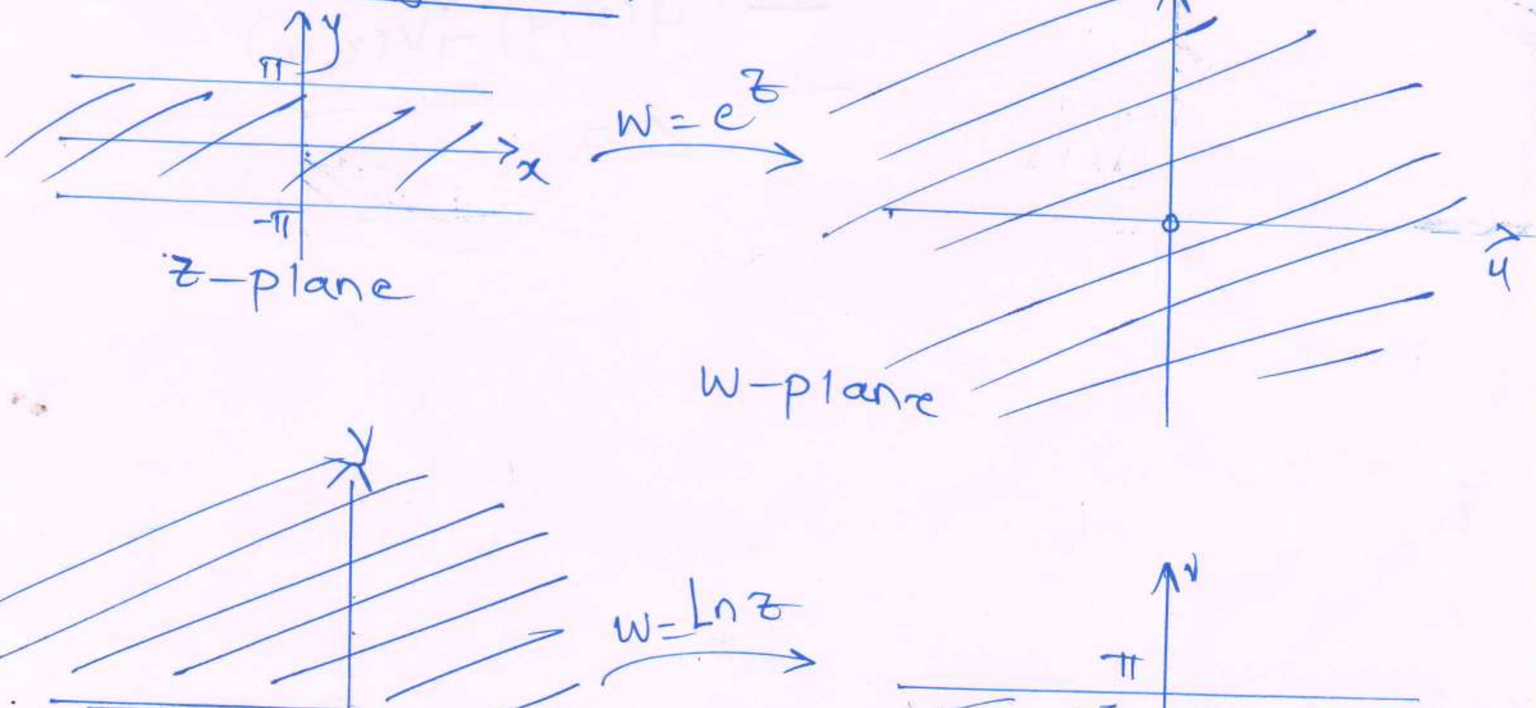
Principle of inverse mapping

The mapping by the inverse $z = f^{-1}(w)$ of $w = f(z)$ is obtained by interchanging the roles of the z-plan & w-plane in the mapping by $w = f(z)$.

- eg. $w = z^2$ & its inverse $w = \sqrt{z}$ (with $\sqrt{z} > 0$ for $z = x > 0$)



Natural Logarithm:



General Powers

Def. Let $z = x+iy$.

$$z^c := e^{c \ln z} \quad (c \in \mathbb{C},\ z \neq 0)$$

- $\ln z$ infinitely many-valued implies so is z^c.
- $z^c = e^{c \operatorname{Ln} z}$ is the principal value of z^c.
- For $c = 1, 2, 3, \ldots$; z^c is single-valued & identical with n^{th} power of z. Similarly for $c = -1, -2, -3, \ldots$
- If $c = \frac{1}{n}$, $n = 1, 2, 3, \ldots$, then

$$z^c = \sqrt[n]{z} = e^{\frac{1}{n}\ln z}$$ (Unique exponent up to multiples of $\frac{2\pi i}{n}$)

$\Rightarrow$ n distinct values of the n^{th} root. $\frac{2\pi i m}{n}$

- If $c = p/q$, again finitely many distinct values
- If c is irrational or complex, z^c is infinitely many-valued

① * What is i^i?

$$i^i = e^{i \log i} = e^{i(i\frac{\pi}{2} \pm 2n\pi i)} = e^{-\frac{\pi}{2} \mp 2n\pi}.$$

Thus i^i is always real. Its principle value is $e^{-\pi/2}$

② $(1+i)^{-1+i} = \exp((-1+i)\ln(1+i))$

$$= \exp\left((-1+i)\left(\tfrac{1}{2}\ln(2) + i\left(\tfrac{\pi}{4} \pm 2n\pi\right)\right)\right)$$

$$= \exp\left(-\tfrac{1}{2}\ln 2 - \left(\tfrac{\pi}{4} \pm 2n\pi\right) + \ldots\right.$$