TUTORIAL 4: MA 633- PARTITION THEORY

1. For $|q| < |x| < 1$ and $|q| < |y| < 1$, prove the identity

$$\sum_{k=-\infty}^{\infty} \frac{x^k}{1 - yq^k} = \sum_{k=-\infty}^{\infty} \frac{y^k}{1 - xq^k}.$$

without using Ramanujan's ${}_1\psi_1$ summation formula.

2. Evaluate the following series in closed form:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+1)}.$$

4. For $i, j \in \mathbb{N}$, let $y_i \neq y_j$ whenever $i \neq j$. Then show that

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_j}{\lambda}\right)} = \frac{1}{(1-x)(1-xy_1)\cdots(1-xy_j)},$$

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda)} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)},$$

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-z/\lambda^2)} = \frac{1+xyz-x^2yz-xy^2z}{(1-x)(1-y)(1-x^2z)(1-y^2z)}.$$

5. Let $Q_m^{(k,\ell)}(n)$ denote the number of partitions of n into m parts where each part differs from the next by at least k and the smallest parts is $\geq \ell$. Then show that

$$\sum_{n=1}^{\infty} Q_m^{(k,\ell)}(n) q^n = \frac{q^{\ell m + km(m-1)/2}}{(q;q)_m}$$