TUTORIAL 5: MA 633: PARTITION THEORY

1. (i) Using Heine's transformation twice, prove the following q-analogue of Euler's theorem:

$$ {}_2\phi_1\left(a,b;c;z\right) = \frac{(abz/c)_\infty}{(z)_\infty}\,{}_2\phi_1\left(\frac{c}{a},\frac{c}{b};c;\frac{abz}{c}\right). $$

(ii) Prove that

$$ (a)_{n-m} = \frac{(a)_n}{(q^{1-n}/a)_m}\left(\frac{-q}{a}\right)^m q^{\frac{m(m-1)}{2}-nm}. $$

(iii) Use (i) and (ii) to prove the q-analogue of Pfaff-Saalschütz theorem:

$$ {}_3\phi_2\left(a,b,q^{-n};c,\frac{abq^{1-n}}{c};q\right) = \frac{(c/a)_n(c/b)_n}{(c)_n(c/(ab))_n}. $$

2. Prove that

$$ 1+\sum_{k=1}^{\infty}\frac{(1-aq^{4k})(aq^2;q^2)_{k-1}(-q;q^2)_k(-a^2)^k q^{4k^2-k}}{(q^2;q^2)_k(-aq;q^2)_k} $$
$$ = \frac{(aq^2;q^2)_\infty}{(-aq;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}a^k q^{k^2}. $$

(**Hint:** Let $b,c,e,N\to\infty$ in Watson's ${}_8\phi_7$ transformation. Then, in the resulting identity, first replace q by q^2, and then let $d=-q$.)

3. Prove the Göllnitz-Gordon identities using the identity in problem 2 above.

$$ \sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2} = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}, $$
$$ \sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;q^2)_k}q^{k^2+2k} = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty.(q^5;q^8)_\infty}. $$