MA 201 – Mid-sem exam (2017) solutions

INDIAN INSTITUTE OF TECHNOLOGY GANDHINAGAR

A
12 pages

Roll Number: ____________ Name: ____________

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I pledge that in this examination I will neither indulge in unfair means nor accept anyone else doing so.

Date: ____________ Signature of Student: ____________

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	Total
Marks													

(WRITE FROM HERE)

1 (a) $\frac{\log(1+z)}{z^4}$ has a pole of order 3 at $z=0$ since

$$\frac{\log(1+z)}{z^4} = \frac{z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \cdots}{z^4}$$

$$= \frac{1}{z^3} - \frac{1}{2z^2} + \frac{1}{3z} - \frac{1}{4} + \frac{z}{5} - \cdots$$

Principal part : $\frac{1}{z^3} - \frac{1}{2z^2} + \frac{1}{3z}$.

(b) $\frac{\tan z}{z}$. Note that $\lim_{z\to 0}(z-0)\frac{\tan z}{z} = \lim_{z\to 0}\frac{\sin z}{\cos z} = \frac{0}{1} = 0$.

Hence $\frac{\tan z}{z}$ has a removable singularity at $z=0$.

(This can also be seen from the fact that $\frac{\tan z}{z} = \frac{z + \text{higher powers of } z}{z} = 1 + \text{higher powers of } z$)

Since $\lim_{z\to 0}\frac{\tan z}{z} = \lim_{z\to 0}\frac{\sin z}{z} \cdot \lim_{z\to 0}\frac{1}{\cos z} = 1 \cdot \frac{1}{\cos(0)} = 1$,

we define $\frac{\tan z}{z}$ at $z=0$ to be 1.

(c) $\frac{z^2+1}{z(z-1)}$ has a pole of order 1 at $z=0$ since
Simple pole

$$\lim_{z\to 0} z \cdot \frac{z^2+1}{z(z-1)} = \frac{0^2+1}{0-1} = -1$$, a finite quantity

Since this is the residue at $z=0$, the principal part is $\frac{-1}{z}$.

(d) $z^n \sin\left(\frac{1}{z}\right)$, for $n \in \mathbb{N}$ fixed, has an essential singularity at $z=0$ since

$$z^n \sin\left(\frac{1}{z}\right) = z^n\left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots\right)$$

has infinitely many negative powers

(If n is even, then we get a term, after a while, as $\frac{c}{z}$ (c, constant), & then $\frac{c_2}{z^3}, \frac{c_3}{z^5}, \ldots$ (Singular part: $\frac{c_1}{z} + \frac{c_2}{z^3} + \frac{c_3}{z^5} + \cdots$

If n is odd, we get, after a while, a constant term c_1 & then $\frac{c_2}{z^2}, \frac{c_3}{z^4}, \ldots$) (Singular part: $\frac{c_2}{z^2} + \frac{c_3}{z^4} + \cdots$)

(2) (i) $\oint_C \frac{dz}{\cosh z}$ $\quad \cosh z = 0 \Rightarrow \frac{e^z + e^{-z}}{2} = 0$

$$\Rightarrow e^z = -e^{-z} \Rightarrow e^{2z} = -1$$

$$\Rightarrow \cancel{2z = \pm 2\pi n i}$$

$$\Rightarrow \cancel{z = \pm \pi n i}, \cancel{n \in \mathbb{N}}.$$

~~There are poles of $\frac{1}{\cosh z}$. Out of them, $z = \pm\pi i$ lie~~

$$e^{2z} = -1 \text{ when } 2z = \pm\pi i \pm 2n\pi i$$

$$= \pm(2n+1)\pi i$$

$$\Rightarrow z = \pm\frac{(2n+1)\pi i}{2}, n \in \mathbb{N}$$

Out of them, $z = \pm\frac{\pi}{2}i$ lie inside the circle of radius 2 centered at the origin.

These are simple poles of $\frac{1}{\cosh z}$.

By residue theorem,

$$\oint_C \frac{dz}{\cosh z} = 2\pi i\left(\operatorname*{Res}_{z=+\frac{\pi}{2}i} \frac{1}{\cosh z} + \operatorname*{Res}_{z=-\frac{\pi i}{2}} \frac{1}{\cosh z}\right)$$

$$= 2\pi i\left(\left.\frac{1}{\sinh z}\right|_{z=\frac{\pi i}{2}} + \left.\frac{1}{\sinh z}\right|_{z=-\frac{\pi i}{2}}\right)$$

$$= 2\pi i\left(\frac{1}{\sinh\left(\frac{\pi i}{2}\right)} + \frac{1}{\sinh\left(-\frac{\pi i}{2}\right)}\right)$$

$$= 2\pi i\left(\frac{1}{i\sin(\pi/2)} + \frac{1}{i\sin(-\pi/2)}\right) = 2\pi i(-i+i) = 0.$$

$$\Rightarrow \oint_C \frac{dz}{\cosh z} = 0.$$

(ii) $a > b > 0$. Note that

$$I = \int_0^\infty \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx$$

Since the integrand is an even fn of x.

- The integrand has no poles on the real axis.
- degree of the integrand $\overset{=-4}{<} -2$. Let $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

Hence The $f(z)$ has poles at $z = \pm ia, \pm ib$, out of which ia & ib are in UHP, and are simple poles

$$I = \frac{1}{2}\left[-2\pi\left(\operatorname{Im}\left(\operatorname*{Res}_{z=ia} f(z)e^{iz}\right) + \operatorname{Im}\left(\operatorname*{Res}_{z=ib} f(z)e^{iz}\right)\right)\right]$$

$$= \frac{1}{2}\left[-2\pi\left(\operatorname{Im}\left(\lim_{z\to ia} \frac{(z-ia)e^{iz}}{(z-ia)(z+ia)(z^2+b^2)}\right) + \operatorname{Im}\left(\lim_{z\to ib} \frac{(z-ib)e^{iz}}{(z-ib)(z+ib)(z^2+a^2)}\right)\right)\right]$$

$$= \frac{1}{2}\left[-2\pi\left(\operatorname{Im}\left(\frac{e^{-a}}{2ia(b^2-a^2)}\right) + \operatorname{Im}\left(\frac{e^{-b}}{2ib(a^2-b^2)}\right)\right)\right]$$

$$= \frac{1}{2}\left[-2\pi\left(\frac{-e^{-a}}{2a(b^2-a^2)} - \frac{e^{-b}}{2b(a^2-b^2)}\right)\right] = \frac{\pi}{2(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

③ $f(z) = \dfrac{1}{z^2-3z+2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(a) $|z|<1 \Rightarrow |z|<2$ also, i.e. $|\frac{z}{2}|<1$

$\Rightarrow f(z) = \dfrac{-1}{2(1-\frac{z}{2})} + \dfrac{1}{1-z}$

$= \dfrac{-1}{2}\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n.$

(b) $1<|z|<2$:

$1<|z| \Rightarrow |\frac{1}{z}|<1$ & $|z|<2 \Rightarrow |\frac{z}{2}|<1$

$f(z) = \dfrac{-1}{2(1-\frac{z}{2})} - \dfrac{1}{z(1-\frac{1}{z})} = \dfrac{-1}{2}\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n - \dfrac{1}{z}\sum_{n=0}^{\infty}\dfrac{1}{z^n}.$

(c) $|z|>2 \Rightarrow |z|>1$ also

Hence $|\frac{2}{z}|<1$ & $|\frac{1}{z}|<1$

$\Rightarrow f(z) = \dfrac{1}{z(1-\frac{2}{z})} - \dfrac{1}{z(1-\frac{1}{z})}$

$= \dfrac{1}{z}\sum_{n=0}^{\infty}\left(\dfrac{2}{z}\right)^n - \dfrac{1}{z}\sum_{n=0}^{\infty}\left(\dfrac{1}{z}\right)^n.$

(d) $0<|z-1|<1$

$\Rightarrow f(z) = \dfrac{-1}{1-(z-1)} - \dfrac{1}{z-1}$

$= \dfrac{-1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n.$

(4) (i) A non-constant entire function is unbounded (by Liouville's thm).

Hence if this unbounded entire function is a poly; then it has a pole at infinity and not an essential singularity because:

if $f(z) = \sum_{n=0}^{k} a_n z^n$, then

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{k} \frac{a_n}{z^n} = a_0 + \frac{a_1}{z} + \cdots + \frac{a_k}{z^k}$$

and we know that $f(z)$ having a pole at infinity is equivalent to $f(\frac{1}{z})$ having a pole at $z=0$.

Note that if $f(\frac{1}{z})$ has a singularity at $z=0$, it has only finite many negative powers (being a polynomial in $1/z$).

Hence it cannot be essential singularity & so it has to be a pole.

If $f(z)$ is not a polynomial, then it is something like e^z, $\sin z$, $\cos z$ etc.

Then since we know that $e^{1/z}$, $\sin(1/z)$ & $\cos(\frac{1}{z})$ etc. have essential singularity at $z=0$, e^z, $\sin z$ or $\cos z$ have essential singularity at $z=\infty$. It cannot be a pole, since, for example,

$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ has infinitely many negative powers of z.

(ii) Let z_0 be a zero of order m of the analytic function $f(z)$.

Then $f(z) = (z-z_0)^m \left(a_m + a_{m+1}(z-z_0) + \cdots\right)$

with $a_m \neq 0$,

Now $a_m + a_{m+1}(z-z_0) + \cdots$ being a power series in $z - z_0$ represente an analytic fn., say $g(z)$.

Note that $g(z_0) = a_m \neq 0$,

Since g, being analytic, is continuous, $\exists$ a neighbor-hood around z_0 on which g is non-zero.

But $(z-z_0)^m$ is zero only for $z = z_0$.

Hence $\exists$ a neighborhood around z_0 on which $f(z)$ is non-zero.

$\Rightarrow$ zeros of f are isolated,

(5) $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-t^2}\,dt$

$$= \frac{2}{\sqrt{\pi}} \int_0^z \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right) dt$$

$$= \frac{2}{\sqrt{\pi}} \left[t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \cdots \right]_0^z$$

$$= \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2!} - \frac{z^7}{7 \cdot 3!} + \cdots \right)$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)}.$$

(6) $\sum_{n=0}^{\infty} z^n q^{n^2}$. (power series in z) $\quad a_n = q^{n^2}$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |q^{n^2}|^{1/n} = \lim_{n\to\infty} |q|^n$$

$$= \begin{cases} 0, & \text{if } |q|<1 \\ 1, & \text{if } |q|=1 \\ \infty, & \text{if } |q|>1 \end{cases}$$

$$\Rightarrow R = \begin{cases} \infty, & \text{if } |q|<1 \\ 1, & \text{if } |q|=1 \\ 0, & \text{if } |q|>1 \end{cases}.$$

Thus as a fn of z, $\sum_{n=0}^{\infty} z^n q^{n^2}$ converges everywhere if $|q|<1$, and converges no-where if $|q|>1$.

(7) z is a non-~~real~~ real complex number

First we prove (i) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

(ii) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

Proof :- we know that

(i) $\cos z = \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$\Rightarrow |\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x (1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y$

$= \cos^2 x + \sinh^2 y$

(ii) $\sin z = \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$

$= \sin x \cosh y + i \cos x \sinh y$

$\Rightarrow |\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x (1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$

$= \sin^2 x + \sinh^2 y$

From (i) & (ii), we find that $|\cos z|^2$ & $|\sin z|^2$ (and hence $|\cos z|$ & $|\sin z|$) tend to infinity as $z \to \infty$ (and hence $y \to \infty$) since $\sinh y \to \infty$ as $y \to \infty$.

Thus $\sin z$ & $\cos z$ are unbounded when z is a non-real complex number.

(8) (i) $i^{i} = e^{i \log i} = \cancel{e^{i(\log}}\ e^{i\left(\frac{i\pi}{2} \pm 2n\pi i\right)}$ $(n \in \mathbb{N})$

$= e^{-\frac{\pi}{2} \mp 2n\pi i}$ $(n \in \mathbb{N})$

This is the general value of i^i.

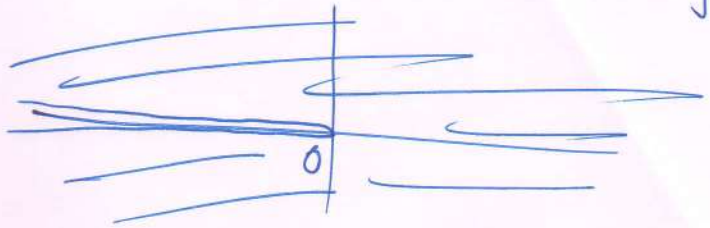
Principal value is when $n = 0$, i.e. $e^{-\pi/2}$

(ii) e^z is entire fn. of z. & has anti-derivative as e^z itself.

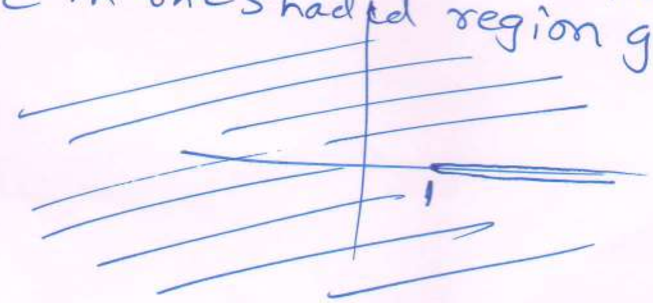
Hence $\int_C e^z = \int_{\frac{\pi i}{2}}^{\pi i} e^z\, dz = \left[e^z\right]_{\frac{\pi i}{2}}^{\pi i} = e^{\pi i} - e^{\frac{\pi i}{2}}$

$= -1 - i.$

(9) We know that $Ln(z)$ (principal value of log) has a branch cut from $(-\infty, 0]$, that ~~...~~ $Ln(z)$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ (shaded region).

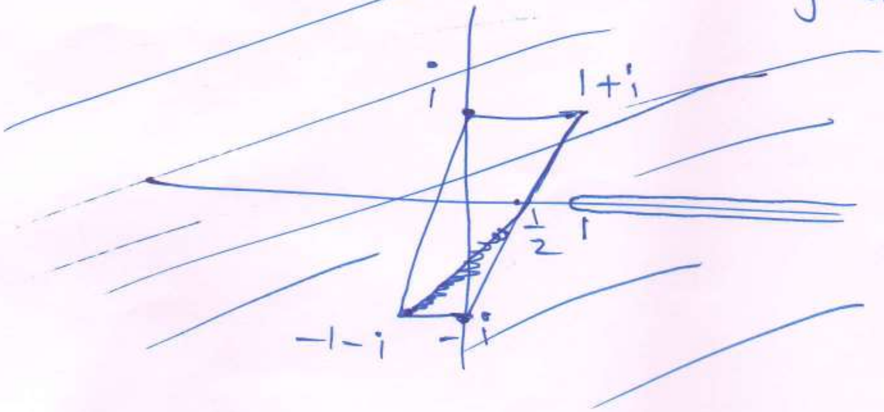


So $Ln(1-z)$ has a branch cut from $[1, \infty)$ & is analytic in the shaded region given below.



$\left(\because z \notin (-\infty, 0] \Rightarrow -z \notin [0, \infty) \Rightarrow 1-z \notin [1, \infty) \right)$

Now the parallelogram with vertices $\pm i$, $\pm(1+i)$ lies in the above shaded region as shown below



Note that line passing through $(0,-1)$ & $(1,1)$ is given by

$$\frac{y+1}{x} = \frac{2}{1}$$

$$\Rightarrow y = 2x - 1$$

So it intersects x axis at $x = \frac{1}{2}$

Since this (simple closed curve) parallelogram lies in the shaded region where $Ln(1-z)$ is analytic, $\oint_C Ln(1-z) = 0$ by Cauchy's integral thm.