ATUL DIXIT

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Homework # 3.

1] Calculate the following integrals :-

a) $\int_0^\infty \frac{x^2\,dx}{x^4+x^2+1}$

Sol^n:- Here deg. $r(x) = -2$ where $r(x) = \frac{x^2}{x^4+x^2+1}$

Also then $r(z) = \frac{z^2}{z^4+z^2+1}$

$\therefore$ P_r = zero set of z^4+z^2+1

So roots of $z^4+z^2+1=0$ can be found as.

$z^4+z^2+1=0$

$\therefore z^4+2z^2+1-z^2=0$

$\therefore (z^2+1)^2 = z^2$

$\therefore z^2+1 = \pm z$

$\therefore z^2+z+1=0$ or $z^2-z+1=0$

$\therefore z = \frac{-1\pm\sqrt{3}i}{2}$ or $z = \frac{1\pm\sqrt{3}i}{2}$

Hence the zero set of $q(z)$ and hence

$P_r = \left\{ \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2} \right\}$

and none of them is on the real axis.

Since only $\frac{-1+\sqrt{3}i}{2}$ and $\frac{1+\sqrt{3}i}{2}$ lie on

UHP, we have that,

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$$\int_0^\infty r(x)\,dx = \pi i \sum_{\substack{a_k \in UHP \\ a_k \in P_r}} \mathrm{Res}\,(r(z); a_k)$$

$$= \pi i \left[\mathrm{Res.}\left(\frac{z^2}{z^4+z^2+1}; \frac{1+\sqrt{3}\,i}{2}\right) + \mathrm{Res}\left(\frac{z^2}{z^4+z^2+1}; \frac{-1+\sqrt{3}\,i}{2}\right) \right]$$

$$= \pi i \left[\mathrm{Res}\left(\frac{z^2}{4z^3+2z}; \frac{1+\sqrt{3}\,i}{2}\right) + \mathrm{Res}\left(\frac{z^2}{4z^3+2z}; \frac{-1+\sqrt{3}\,i}{2}\right) \right]$$

$$= \pi i \left[\mathrm{Res}\left(\frac{z}{4z^2+2}; \frac{1+\sqrt{3}\,i}{2}\right) + \mathrm{Res}\left(\frac{z}{4z^2+2}; \frac{-1+\sqrt{3}\,i}{2}\right) \right]$$

$$= \pi i \left[\frac{\frac{1+\sqrt{3}\,i}{2}}{4\left(\frac{1}{4} - \frac{3}{4} + \frac{\sqrt{3}}{2} i\right) + 2} + \frac{\frac{-1+\sqrt{3}\,i}{2}}{4\left(\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2} i\right) + 2} \right]$$

$$= \pi i \left[\frac{\frac{1+\sqrt{3}\,i}{2}}{-2+2+2\sqrt{3}\,i} + \frac{\frac{-1+\sqrt{3}\,i}{2}}{-2+2-2\sqrt{3}\,i} \right]$$

$$= \frac{\pi i}{2\sqrt{3}\,i} \left[\frac{1}{2} + \frac{\sqrt{3}}{2} i + \frac{1}{2} - \frac{\sqrt{3}}{2} i \right]$$

$$= \frac{\pi}{2\sqrt{3}}(1)$$

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$$= \frac{\pi\sqrt{3}}{6}$$

$$\therefore \int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx = \frac{\pi\sqrt{3}}{6}$$

(c) (b) $\cancel{\int_0^\infty \frac{\cos x - 1}{x^2}\,dx}$ $\int_0^\pi \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}$, where $a^2 < \cancel{1}$

Sol^n:- Let $I = \int_0^\pi \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}$

why do this?

$$= \int_0^\pi \frac{\cos 2\theta\,(1+2a\cos\theta+a^2)}{(1-2a\cos\theta+a^2)(1+2a\cos\theta+a^2)}\,d\theta$$

$$= \int_0^\pi \frac{(1+a^2)\cos 2\theta + 2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta.$$

$$= \int_0^\pi \frac{(1+a^2)\cos 2\theta\, d\theta}{(1+a^2)^2 - 4a^2\cos^2\theta} + 2a\int_0^\pi \frac{\cos 2\theta\cos\theta\, d\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}$$

Now $\int_0^\pi \frac{\cos 2\theta\cos\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta = 0$

(because if $f(\theta) = \frac{\cos 2\theta\cos\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}$)

then $f(\pi-\theta) = \frac{\cos(2(\pi-\theta))\cos(\pi-\theta)}{(1+a^2)^2 - 4a^2(\cos(\pi-\theta))^2}$

$$= \frac{-\cos(2\theta)\cos\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}$$

$$= -f(\theta)$$

~~and we know that~~ $\cancel{\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx}$

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and we know that $\int_0^{\pi} f(\theta)\,d\theta = \int_0^{\pi} f(\pi-\theta)\,d\theta$.

So if $f(\pi-\theta) = -f(\theta)$,

we have $\int_0^{\pi} f(\theta)\,d\theta = -\int_0^{\pi} f(\theta)\,d\theta$

$\therefore 2\int_0^{\pi} f(\theta)\,d\theta = 0$

$\therefore \int_0^{\pi} f(\theta)\,d\theta = 0$)

Hence $\int_0^{\pi} \frac{\cos 2\theta \cos\theta \, d\theta}{(1+a^2)^2 - 4a^2\cos^2\theta} = 0$.

$\therefore I = \int_0^{\pi} \frac{(1+a^2)\cos 2\theta \, d\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}$.

Let $2\theta = t$

$\therefore d\theta = dt/2$

when $\theta = 0$ $t = 0$

when $\theta = \pi$, $t = 2\pi$

$\therefore I = \int_0^{2\pi} \frac{(1+a^2)\cos t \, \frac{dt}{2}}{(1+a^2)^2 - 2a^2(1+\cos t)}$

$= \frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos t \, dt}{1+a^4 - 2a^2\cos t}$

Let $z = e^{it}$ $\therefore dz = ie^{it}\,dt$

$\therefore dt = \frac{dz}{iz}$ Also $\cos t = \frac{z + \frac{1}{z}}{2}$ $\left(\because \bar{z} = \frac{1}{z}\right)$

$$\therefore I = \frac{1}{2}\int_{|z|=1} \frac{(1+a^2)\left(z+\frac{1}{z}\right)\cdot\frac{1}{2}}{(1+a^4) - 2a^2\left(z+\frac{1}{z}\right)\cdot\frac{1}{2}} \frac{dz}{iz}$$

$$= \frac{(1+a^2)}{4i}\int_{|z|=1} \frac{\left(z+1/z\right)\,dz}{z\left[(1+a^4) - a^2\left(z+\frac{1}{z}\right)\right]}$$

$$= \frac{1+a^2}{4i}\int_{|z|=1} \frac{(z^2+1)\,dz}{z\left[(1+a^4)z - a^2z^2 - a^2\right]}$$

$$= \frac{1+a^2}{4i}\int_{|z|=1} \frac{(z^2+1)\,dz}{z\left[-a^2z^2 + (a^4+1)z - a^2\right]}$$

$$= \frac{1+a^2}{(-a^2)\,4i}\int_{|z|=1} \frac{(z^2+1)\,dz}{z\left[z^2 - \left(a^2+\frac{1}{a^2}\right)z + 1\right]}$$

$$= \frac{(1+a^2)i}{a^2\,4}\int_{|z|=1} \frac{(z^2+1)\,dz}{z\,(z-a^2)\left(z-\frac{1}{a^2}\right)}$$

$$= \frac{(a^2+1)i}{a^2\,4}\int_C f(z)\,dz$$

where $f(z) = \dfrac{z^2+1}{z(z-a^2)\left(z-\frac{1}{a^2}\right)}$

Now $f(z)$ has simple poles at $z=0, a^2$, and $\frac{1}{a^2}$, out of which $z=0$ and $z=a^2$ lie inside $|z|=1$ $\left(\because a^2<1\right)$

$$\therefore I = \frac{(a^2+1)\, i \cdot 2\pi i}{4a^2}\left[\operatorname{Res}\left(f(z); 0\right) + \operatorname{Res}\left(f(z); a^2\right)\right]$$

$$= \frac{(a^2+1)\, i \cdot 2\pi i}{4a^2}\left[\lim_{z\to 0} z f(z) + \lim_{z\to a^2}(z-a^2) f(z)\right]$$

$$= \frac{(a^2+1)\, i \cdot 2\pi i}{4a^2}\left[\lim_{z\to 0} \frac{z^2+1}{(z-a^2)\left(z-\frac{1}{a^2}\right)} + \lim_{z\to a^2} \frac{z^2+1}{z\left(z-\frac{1}{a^2}\right)}\right]$$

$$= \frac{(a^2+1)\, i \cdot 2\pi i}{4a^2}\left[1 + \frac{(a^4+1)\cdot a^2}{a^2(a^4-1)}\right]$$

$$\cancel{= \frac{(a^2+1)\, i}{4a^2}\left[1 + \frac{1}{a^2+1}\right]}$$

$$= \frac{(a^2+1)i \cdot 2\pi i}{4a^2}\left[1 + \frac{a^4+1}{a^4-1}\right]$$

$$= \frac{(a^2+1)\, 2\pi i^2}{4a^2}\left[\frac{2a^4}{a^4-1}\right]$$

$$= \frac{-\pi a^2}{(a^2-1)} = \frac{\pi a^2}{1-a^2}$$

$$\therefore \int_0^{\pi} \frac{\cos\theta\, d\theta}{1-2a\cos\theta + a^2} = \frac{\pi a^2}{1-a^2} \quad \left(a^2<1\right)$$

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Generalization of Casorati-Weierstrass theorem

Prob. 10 - Ex. 5.1 - pg. 110 (CONWAY)

* Suppose that if f has an essential singularity at $z=a$. If $c \in \mathbb{C}$ and $\varepsilon > 0$ are given, then prove that for each $\delta > 0$, there is a number α, $|c-\alpha| < \varepsilon$, such that $f(z) = \alpha$ has infinitely many solutions in $B(a;\delta)$.

Proof:- Consider $A_n = f\left(B\left(a, \frac{1}{n}\right)'\right)$ $n = 1, 2, 3, \ldots$

where $B\left(a, \frac{1}{n}\right)'$ denotes an open ball of radius $1/n$, punctured at $z=a$.

Since f has an essential singularity at $z=a$, by the simple version of Casorati-Weierstrass theorem that $\overline{f\left(B(a,\frac{1}{n})'\right)} = \mathbb{C}$ for each $n \in \mathbb{N}$.

That is, each A_n is dense in $\mathbb{C}$ — (1)

Also since f is analytic in $B(a,\frac{1}{n})'$, $f(B(a,\frac{1}{n})')$ is open for each $n \in \mathbb{N}$ (because analytic function is an open map).

Thus each A_n is open in $\mathbb{C}$ — (2)

Now from (1), (2) and Baire Category theorem, we have that $\bigcap_{n=1}^{\infty} A_n$ is dense in $\mathbb{C}$

(Baire category theorem states that in a complete metric space (or compact Hausdorff space) if $\{U_n\}_{n=1}^{\infty}$ is a collection of open dense sets in X, then

$\bigcap_{n=1}^{\infty} U_n$ is dense in X)

But we know that if A is a set dense in X, then A intersects every non-empty open set in X.

Hence consider $B(c;\varepsilon)$ where $\varepsilon > 0$.

Let $\alpha \in B(c;\varepsilon) \cap \left(\bigcap_{n=1}^{\infty} A_n\right)$

But then $\alpha \in A_n$ for each $n \in \mathbb{N}$

i.e. $\alpha \in f\left(B(a,\frac{1}{n})'\right)$ for each $n \in \mathbb{N}$

which means that there is a solution of α in each of the sets $f\left(B(a,\frac{1}{n})'\right)$, which means that there are infinitely many solutions of $f(z) = \alpha$ in $B(a;\delta)$ (where n is chosen in the first place such that given $\delta > 0$, $\frac{1}{n} < \delta$).

Hence proved

6] Let γ be ~~a~~ the rectangular path $[n+\frac{1}{2}+ni, -n-\frac{1}{2}+ni, -n-\frac{1}{2}-ni, n+\frac{1}{2}-ni, n+\frac{1}{2}+ni]$ and evaluate the integral $\int_\gamma \pi(z+a)^{-2}\cot\pi z\,dz$ for $a \neq$ an integer. Show that $\lim_{n\to\infty} \pi(z+a)^{-2}\cot\pi z\,dz = 0$, and by using the first part, deduce that,

$$\frac{\pi^2}{\sin^2\pi a} = \sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}$$

Proof :- $\gamma = [n+\frac{1}{2}+ni, -n-\frac{1}{2}+ni, -n-\frac{1}{2}-ni, n+\frac{1}{2}-ni, n+\frac{1}{2}+ni]$

Now $I = \int_\gamma \frac{\pi}{(z+a)^2}\cot\pi z\,dz = \int_\gamma \frac{\pi \cos\pi z\,dz}{(z+a)^2\sin\pi z}$

Now let $f(z) = \frac{\pi\cos\pi z}{(z+a)^2\sin\pi z}$.

f has poles ~~of~~ order 2 at $z = -a$ and pole of order 1 at ~~all~~ every integer.

$\therefore I = 2\pi i$ (sum of residues at poles which lie inside γ)

(by Cauchy's residue theorem)

Now ~~since~~ according to defn. of γ, No. of poles contained in γ are 1 to n and $-n$ to -1 and pole of order 2 at $z = -a$

~~Hence $I = 2\pi$~~

Now residue at a pole of order 1 at $z=m$ is

$$\left.\frac{\pi\cos\pi z}{\frac{d}{dz}((z+a)^2\sin\pi z)}\right|_{z=m}$$

$$=\left.\frac{\pi\cos\pi z}{\pi(z+a)^2\cos\pi z+2(z+a)\sin\pi z}\right|_{z=m}$$

$$=\frac{\pi\cos m\pi}{\pi(a+m)^2\cos m\pi+2(m+a)\sin m\pi}$$

$$=\frac{\pi\cos m\pi}{\pi(a+m)^2\cos m\pi+0} \quad (\because \sin m\pi=0 \ \forall m\in\mathbb{Z})$$

$$=\frac{1}{(a+m)^2}$$

Also residue at pole of order 2 at $z=-a$ is

$$\left.\frac{1}{(2-1)!}\frac{d}{dz}(z+a)^2 f(z)\right|_{z=-a}$$

$$=\left.\frac{d}{dz}\frac{(z+a)^2\pi\cos\pi z}{(z+a)^2\sin\pi z}\right|_{z=-a}$$

$$=\left.\pi\frac{d}{dz}\cot\pi z\right|_{z=-a}$$

$$=\left.\pi^2(-\operatorname{cosec}^2\pi z)\right|_{z=-a}$$

$$=\frac{-\pi^2}{\sin^2\pi(-a)}=\frac{-\pi^2}{\sin^2\pi a} \quad \left(\text{and } \sin\pi a\neq 0 \ \because a\notin\mathbb{Z}\right)$$

Hence $\int_{\gamma} \frac{\pi}{(z+a)^2} \cot \pi z \, dz = 2\pi i \left\{ \sum_{m=-n}^{n} \frac{1}{(a+m)^2} - \frac{\pi^2}{\sin^2 \pi a} \right\}$

Thus

$$\lim_{n\to\infty} \int_{\gamma} \frac{\pi \cot \pi z}{(z+a)^2} \, dz = 2\pi i \left(\sum_{m=-\infty}^{\infty} \frac{1}{(a+m)^2} - \frac{\pi^2}{\sin^2 \pi a} \right) \quad (1)$$

Now we will show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$

$|\cos z|^2 = |\cos(x+iy)|^2$

$= |\cos x \cosh y - i \sin x \sinh y|^2$

$= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x (1 + \sinh^2 y) + (1 - \cos^2 x) \sinh^2 y$

$= \cos^2 x + \sinh^2 y.$

Also $|\sin z|^2 = |\sin(x+iy)|^2$

$= |\sin x \cosh y + i \cos x \sinh y|^2$

$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x (1 + \sinh^2 y) + (1 - \sin^2 x) \sinh^2 y$

$= \sin^2 x + \sinh^2 y.$

$\therefore |\cos \pi z|^2 = \cos^2 \pi x + \sinh^2 \pi y$

and $|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y$

$\therefore |\cot \pi z|^2 = |\cot^2 (\pi z)| = \left| \frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y} \right|$

Now when $y \to \infty$, (since γ becomes very large), $\sinh^2 \pi y \to \infty$ (because $\sinh^2 \pi y = \frac{(e^{\pi y} - e^{-\pi y})}{2}$). Hence $|\cot^2 \pi z| =$

$$\left| \frac{\frac{\cos^2 \pi x}{\sinh^2 \pi y} + 1}{\frac{\sin^2 \pi x}{\sinh^2 \pi y} + 1} \right| \to 1 \quad \text{as } y \to \infty \qquad \text{--- (1)}$$

Also as $x \to \infty$, we know that for z on the vertical ~~strip~~ segments, $\cos \pi x = 0$ and $\sin \pi x = 1$ ($\because$ the vertical ~~strip~~ segment is passing through $x = p + \frac{1}{2}$ for $p \in \mathbb{N}$, $\cos \pi x = \cos(\pi p + \frac{\pi}{2}) = 0$ and $\sin \pi x = \sin(\pi p + \frac{\pi}{2}) = (-1)^p \therefore \sin^2(\pi p + \pi/2) = 1$)

$$\text{Hence } |\cot^2 \pi z| = \left| \frac{0 + \sinh^2 \pi y}{1 + \sinh^2 \pi y} \right| \text{ as } x \to \infty$$

$$= \left| \frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y} \right| < 1 \qquad \text{--- (2)}$$

Hence from (1) and (2), ~~wh ca~~ when n becomes sufficiently large, i.e. when x and y become sufficiently large, we can have them bounded above by 4 ($\because$ ~~but~~ $\cot^2 \pi z \to 1$ when $x \to \infty, y \to \infty$). Hence $|\cot^2 \pi z| \leq 4$

Hence $|\cot \pi z| \leq 2$ for z on γ if n is sufficiently large.

$\therefore$ As $n \to \infty$

$$\left| \int_\gamma \frac{\pi}{(z+a)^2} \cot \pi z \, dz \right| \leq \max_\gamma |\cot \pi z| \cdot \frac{\pi}{|z+a|^2} \cdot l(\gamma)$$

(where $l(\gamma)$ is length of rectangle)

$$\leq \frac{2\pi \cdot l(\gamma)}{|z+a|^2} \xrightarrow{\text{as}} 0$$

$|z| \to \infty$ (as $n \to \infty$)

hence $|z+a|^2 \to \infty$,

Hence $\left| \int_\gamma \frac{\pi}{(z+a)^2} \cot \pi z \, dz \right| \leq 0$

But absolute value cannot be less than zero

Hence $\int_\gamma \frac{\pi}{(z+a)^2} \cot \pi z = 0$ as $n \to \infty$

i.e. $\lim_{n \to \infty} \int_\gamma \frac{\pi}{(z+a)^2} \cot \pi z = 0$.

But then from ①,

$$2\pi i \left[\sum_{m=-\infty}^{\infty} \frac{1}{(a+m)^2} - \frac{\pi^2}{\sin^2 \pi a} \right] = 0$$

$$\therefore \sum_{m=-\infty}^{\infty} \frac{1}{(a+m)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

i.e. $\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2 \pi a}$ 4

Hence proved.