MA 502 - Tutorial 13 Solutions

(1) $p(z) = z^4 + z - 7$

Let $\gamma_1 : |z| = 1$, let $f(z) = -7$ and $g(z) = z^4 + z$.
Then,

$$0 < |g(z)| = |z^4 + z| \le |z|^4 + |z| = 2 < |-7| = |f(z)| < \infty$$

So $p(z)$ has the same roots as $f(z) = -7$ in $|z| < 1$.

$\Rightarrow$ $p(z)$ has no roots inside $\gamma_1 : |z| = 1$.

Now let $\gamma_2 : |z| = 2$, $f(z) = z^4$, $g(z) = z - 7$.
Then

$$0 < |g(z)| = |z - 7| \le |z| + 7 = 9 < |z|^4 = 16 = |f(z)|$$

$\Rightarrow$ $p(z)$ has the same number of roots as z^4 inside $|z| = 2$.

But z^4 has 4 zeros (all $= 0$) inside $|z| = 2$ by FTA. Moreover, $p(z)$ also has 4 roots. Hence all 4 roots of p lie in $\text{ann}(0; 1, 2)$.

(2) $\alpha z e^z = 1 \Rightarrow \alpha z = e^{-z}$ ($\because e^z \neq 0$ for any $z \in \mathbb{C}$)

$\Rightarrow \alpha z - e^{-z} = 0$

Let $f(z) = \alpha z$ & $g(z) = -e^{-z}$

Let $\gamma : |z| = 1$. Then

$$0 < |g(z)| = |-e^{-z}| = e^{-x} \le e^1 < |\alpha z| < \infty$$

$(x = Re(z))$; $\because (-1 < x < 1)$; $(\because |\alpha| > e$ & $|z| = 1)$

Hence $\alpha z - e^{-z}$ has the same number of roots as αz in $B(0;1)$. and that is
But αz has only one root in $B(0;1)$, namely, the root $z = 0$.

Hence proved.

(3) A is a set dense in $X \Rightarrow \bar{A} = X$.
To prove: A intersects every non-empty open set in X.

Suppose $\exists\, U \neq \phi$ open in X with $A \cap U = \phi$
Then $A \subseteq U^c$.
But since U is open, U^c is closed.
$\Rightarrow U^c$ is a closed set containing X, so it must contain $\bar{A}$.
$\Rightarrow X \subseteq U^c$ But $U^c \subseteq X \Rightarrow U^c = X$
$\Rightarrow U = \phi$ $\rightarrow\leftarrow$
Hence $A \cap U \neq \phi$

Actually the converse is true as well, for, if A intersects every non-empty open set in X and $\exists\, x \in X \ni x \notin \bar{A}$.
Then $x \in \bar{A}^c$, which is an open set so that
$A \cap \bar{A}^c \neq \phi$.
But this is absurd since $A \subseteq \bar{A}$.
$\Rightarrow \bar{A} = X$.

$\boxtimes$

(5) W.l.g. we can assume f to have only one pole, say of order m, at $z=a$.

Then $f(z) = \frac{g(z)}{(z-a)^m}$, where $g(z)$ is analytic in G with $g(a) \neq 0$.

Then we define $f: G \to \mathbb{C}_\infty$ by

$$f(z) = \begin{cases} \frac{g(z)}{(z-a)^m}, & z \neq a \\ \infty, & z = a. \end{cases}$$

Note that f is obviously continuous on $\mathbb{C} \setminus \{a$

Moreover,

$$\lim_{z \to a} f(z) = \lim_{z \to a} \frac{g(z)}{(z-a)^m} = \infty = f(a).$$

↑ Note that $g(a) \neq 0$

Hence f is continuous at $z=a$ as well.

$\boxtimes$

5.3 pg.126)

2] Suppose f is analytic on $\bar{B}(0;1)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions (counting multiplicities) of the eqn. $f(z) = z^n$ where n is an integer larger than or equal to 1.

Soln:- We use Rouche's theorem (second version). So we show that if $G(z) = f(z) - z^n$ and $F(z) = 2z^n$, then on $\partial B(0;1)$, $0 < |G(z)| < |F(z)| < \infty$ and so by Rouche's theorem #2, F and $F+G$ has same number of roots in $B(0;1)$.

Now by reverse triangle inequality,

$$\big|\,|-f(z)| - |z^n|\,\big| \le |-f(z) - z^n|$$

But $\big|\,|-f(z)| - |z^n|\,\big| = |k - 1| > 0$ (strictly greater, for otherwise, $|f(z)| = |z^n|$ which is not true since $|f(z)| < 1 = |z|^n$.)

(where $|-f(z)| = k$, say)

$\therefore 0 < |G(z)|$

Also
$$|G(z)| = |-f(z) - z^n|$$
$$= |f(z) + z^n|$$
$$\le |f(z)| + |z^n|$$
$$< 1 + 1$$

($\because |f(z)| < 1$ when $|z| = 1$ and $\because |z| = 1$, $|z|^n = |z^n| = 1$)

$= 2 = |2z^n| = |F(z)|$ $\left(\because |z|^n = |z^n| = 1 \right)$

$\therefore |G(z)| < |F(z)|$

Also clearly $|F(z)| < \infty$ $\left(\because |F(z)| = |2z^n| = 2 \right)$

$\therefore 0 < |G(z)| < |F(z)| < \infty$

Hence $2z^n$ and $-f(z) - z^n + 2z^n$ have same number of roots.

But $2z^n$ has n number of roots (i.e. root at $z = 0$ of multiplicity n) ~~inside~~ in $B(0;1)$

Hence $-f(z) - z^n + 2z^n = -f(z) + z^n$ have 'n' number of roots.

Thus $-f(z) + z^n = 0$ i.e. $f(z) = z^n$ have n number of roots.

4