MA 502 - Tutorial 9 (Complex Analysis)

1. Suppose $f : G \to \mathbb{C}$ is analytic and define $\varphi : G \times G \to \mathbb{C}$ by

$$\varphi(z,w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{if } z \neq w, \\ f'(z) & \text{if } z = w. \end{cases}$$

Prove that φ is continuous and for each fixed w, the function $z \to \phi(z,w)$ is analytic.

2. Let $B_{\pm} = \overline{B}\left(\pm 1; \frac{1}{2}\right), G = B(0;3) - (B_+ \cup B_-)$. Let γ_1, γ_2 and γ_3 be curves whose traces are $|z-1| = 1, |z+1| = 1$ and $|z| = 2$ respectively. Give γ_1, γ_2 and γ_3 orientations such that $n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0$ for all w in $\mathbb{C} \backslash G$.

3. Prove that Cauchy's integral formula

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(z)\, dz}{z - a}$$

follows from Cauchy's theorem

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0,$$

where the hypotheses for both the results are as discussed before.

4. Let f be analytic on $B(0;1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1$.

5. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Find $\displaystyle\int_{\gamma} \left(\frac{z}{z-1} \right)^n dz$ for all positive integers n.

6. Let $G = \mathbb{C} \backslash \{0\}$. Show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.

7. (a) Let G be a region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

(b) Use (a) to show that the Riemann zeta function $\zeta(s)$ defined for $\text{Re}(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is analytic in $\text{Re}(s) > 1$.