MA-509: HOMEWORK 1 (DUE SEPTEMBER 18)

1. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of* y *to the base* b.)

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for sufficiently large n; to see this apply part (c) with $t = yb^{-w}$.

(e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup(A)$ satisfies $b^x = y$.

(g) Prove that this x is unique.

2. Let $A_1, A_2, A_3, \cdots$ be subsets of a metric space.

(a) If $B_n = \cup_{i=1}^n A_i$, prove that $\overline{B_n} = \cup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, \ldots$

(b) If $B = \cup_{i=1}^\infty A_i$, prove that $\cup_{i=1}^\infty \overline{A_i} \subset \overline{B}$. Also show by means of an example, that this inclusion can be proper.

3. Construct a bounded set of real numbers with exactly three limit points.