MA-509: HOMEWORK 2 (DUE NOVEMBER 20)

1. Prove that there is no value of k such that $x^3 - 3x + k = 0$ has 2 distinct roots in the closed interval $[0, 1]$.

2. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0, \text{if } x \text{ is irrational,} \\ 1/n, \text{if } x \text{ is } m/n. \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

3. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)