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MA 509 - REAL ANALYSIS — LECTURE 10

METRIC SPACES

Defn. A metric space X is a set such that with any two elements p, q of X (called points), there is associated a real number $d(p,q)$ (called the distance from p to q) s.t.

(a) $d(p,q) > 0$ if $p \neq q$ & $d(p,p) = 0$.
(b) $d(p,q) = d(q,p)$
(c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

(d is called the distance function or metric.)

Eg: ① $\mathbb{R}$ is a metric space.

② $\mathbb{R}^k$ (Euclidean space) with the metric $d(\bar{x}-\bar{y}) = |\bar{x}-\bar{y}|$, $(x, y \in \mathbb{R}^k)$.

③ $\mathcal{C}(X) :=$ the set of all complex-valued continuous, bounded functions with domain X.
Associated distance fn. $\|f-g\|$ if $f, g \in \mathcal{C}(X)$, where $\|f\| = \sup_{x \in X} |f(x)|$. (supremum norm)

④ $\mathcal{L}^2(\mu) = \{f : f$ is measurable on a space X, and if $\int_X |f|^2 d\mu < \infty\}$

with $\|f\| = \left(\int_X |f|^2 d\mu\right)^{1/2}$. ($\mathcal{L}^2(\mu)$ norm of f).

- Every subset of a metric space is a metric space.

Some definitions:

① Segment $(a,b) := \{x : x \in \mathbb{R},\ a < x < b\}$.

② Interval $[a,b] := \{x : x \in \mathbb{R},\ a \le x \le b\}$.

③ Half-open intervals:
$[a,b) := \{x \in \mathbb{R} : a \le x < b\}$, $(a,b] := \{x \in \mathbb{R},\ a < x \le b\}$.

④ k-cell $= \{\bar{x} = (x_1, x_2, \dots, x_k) : \bar{x} \in \mathbb{R}^k,\ a_i \le x_i \le b_i,\ 1 \le i \le k\}$.

(1-cell: interval, 2-cell: rectangle)

⑤ Let $\bar{x} \in \mathbb{R}^k$ and $r > 0$.
An open ball B with center at $\bar{x}$ and radius r
$:= \{\bar{y} : \bar{y} \in \mathbb{R}^k,\ |\bar{y} - \bar{x}| < r\}$

Closed ball $:= \{\bar{y} : \bar{y} \in \mathbb{R}^k,\ |\bar{y} - \bar{x}| \le r\}$.

⑥ $E \subseteq \mathbb{R}^k$ convex if $\lambda\bar{x} + (1-\lambda)\bar{y} \in E$ whenever $\bar{x}, \bar{y} \in E$ and $0 < \lambda < 1$.

* Open balls are convex: If $\bar{y}, \bar{z} \in B(\bar{x}, r)$, then for any $\lambda \ni 0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda\bar{y} + (1-\lambda)\bar{z} - \bar{x}| &= |\lambda(\bar{y} - \bar{x}) + (1-\lambda)(\bar{z} - \bar{x})| \\ &\le \lambda|\bar{y} - \bar{x}| + (1-\lambda)|\bar{z} - \bar{x}| \\ &< \lambda r + (1-\lambda) r \\ &= r. \end{aligned}$$

Similarly, closed balls are convex.
k-cells are convex (Ex.)