14/9/2020

MA 509 - REAL ANALYSIS LEC. 13

Thm. 2.12 A set E is open iff its complement is closed.

"$\Leftarrow$"
Proof: Suppose E^c is closed. We want to show that E is open, i.e., every point of E is an interior point of E.

To that end, take a point $x \in E$. So $x \notin E^c$.

Now E^c being closed contain all its limit points. Hence, x cannot be a limit point of E^c.

Therefore, $\exists$ nbhd U of x which does not intersect E^c at all, not even in x (since $x \notin E^c$).

Hence $U \cap E^c = \phi$ implies $U \subset E$ so that x is an interior point of E.

$\Rightarrow$ "E is open".

"$\Rightarrow$" Let x be a limit point of E^c.

Goal: $x \in E^c$

Since x is a limit point of E^c, every nbhd of x intersects E^c in a point other than x.

So no nbhd of x lies in E.

$\Rightarrow$ $x \notin E$ (because if $x \in E$, then E being open would imply that x is an interior point of E $\Rightarrow$ $\exists$ nbhd N of x lying completely in E).

$\Rightarrow$ $x \in E^c$.

$\Rightarrow$ E^c is closed.

Cor. 2.13 A set F is closed iff its complement is open.

Thm. 2.14 (i) If $\{G_\alpha\}$ is any collection of open sets, then $\bigcup_\alpha G_\alpha$ is open.

(ii) If $\{F_\alpha\}$ is any collection of closed sets, then $\bigcap_\alpha F_\alpha$ is closed.

(iii) If $G_1, G_2, \ldots, G_n$ is any finite collection of open sets, $\bigcap_{k=1}^{n} G_k$ is open.

(iv) If $F_1, F_2, \ldots, F_n$ is any finite collection of closed sets, then $\bigcup_{k=1}^{n} F_k$ is closed.

Proof: (i) Each G_α is open. So if $x \in \bigcup_\alpha G_\alpha$.

Claim: x is an interior point of $\bigcup_\alpha G_\alpha$.

If $x \in \bigcup_\alpha G_\alpha$, then $x \in G_\alpha$ for some α.

G_α being open implies that x is an interior point of G_α, and hence an int. pt. of $\bigcup_\alpha G_\alpha$ ∎

(ii) $\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F_\alpha^c$.

F_α is closed $\Rightarrow F_\alpha^c$ is open

So by (i) $\bigcup_\alpha F_\alpha^c$ is open

$\Rightarrow \bigcap_\alpha F_\alpha$ is closed.

(iii) Let $G = \bigcap_{i=1}^{n} G_i$. Let $x \in G$.

Then $x \in G_i \ \forall i \ni 1 \leq i \leq n$.

G_i open $\Rightarrow \exists$ nbhd N_{r_i} of x of radius r $\ni N_{r_i} \subset G_i \ \forall \ 1 \leq i \leq n$.

Now take $r = \min_{1 \leq i \leq n} r_i$. Let N be the nbhd of radius r around x, $\Rightarrow N \subseteq G_i$ for all $1 \leq i \leq n \Rightarrow N \subseteq \bigcap_{i=1}^{n} G_i \Rightarrow \bigcap_{i=1}^{n} G_i$ is open,

(iv) $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$

Examples: In (iii) & (iv), finiteness is absolutely essential, for, if $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$.
Then $\bigcap_{n=1}^{\infty} G_n = \{0\}$ is not open in $\mathbb{R}^1$.

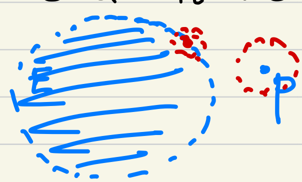
Defn: If X is a metric space, $E \subset X$ and E' denotes the set of all limit points of E in X, then the closure of E is the set $\bar{E} = E \cup E'$.

Eg: ① Let $E = (0,1)$, $X = \mathbb{R}$, then, $\bar{E} = E' = [0,1]$.

② Let $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $X = \mathbb{R}$, then
$E' = \{0\}$, $\bar{E} = E \cup \{0\}$.

Thm. 2.15 If X is a metric space and $E \subset X$, then
(a) $\bar{E}$ is closed.
(b) $E = \bar{E}$ iff E is closed
(c) $\bar{E} \subset F$ for every closed set $F \subset X \ni E \subset F$.

Proof:- (a) If $p \in X$, $p \notin \bar{E}$, then p is neither a point of E nor a limit point of E.
So $\exists$ nbhd N of $p \ni N \cap \bar{E} = \phi$.
$\Rightarrow N \subset \bar{E}^c$ so that p is a interior point of $\bar{E}^c$.
$\Rightarrow \bar{E}^c$ is open.
$\Rightarrow \bar{E}$ is closed.



(b) "$\Rightarrow$" $E = \bar{E}$. By (a), $\bar{E}$ is closed $\Rightarrow E$ is closed
"$\Leftarrow$" If E is closed, $E' \subset E \Rightarrow \bar{E} = E \cup E' = E$.

(c) F is closed, so F contains all its limit points, that $F \supset F'$.

But $F \supset E$. So $F' \supset E'$

$\Rightarrow F \supset E'$.

$\Rightarrow F \supset E \cup E' = \bar{E}$.

<u>Remark</u>: $\bar{E}$ is the <u>smallest</u> closed subset of X that contains E.

<u>Thm. 2.16</u> Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup(E)$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

<u>Proof:</u> If $y \in E$, then clearly $y \in \bar{E}$.
If $y \notin E$, then y is a limit point of E is what we want to show.

Since $y = \sup(E)$, $\exists\ x \in E \ni y - h < x < y$ for $h > 0$, otherwise $y - h$ would be an upper bound of E. But then y must be the limit point of E. $\Rightarrow y \in E' \subset \bar{E}$.

Hence in both the cases, we have $y \in \bar{E}$. $\blacksquare$