16/9/2020

MA 509 - REAL ANALYSIS LEC. 14

LAST TIME:

Thm. 2.15 If X is a metric space and $E \subset X$, then
(a) $\bar{E}$ is closed.
(b) $E = \bar{E}$ iff E is closed
(c) $\bar{E} \subset F$ for every closed set $F \subset X \ni E \subset F$.

Remark: $\bar{E}$ is the smallest closed subset of X that contains E.

Thm. 2.16 Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup(E)$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Proof: If $y \in E$, then clearly $y \in \bar{E}$.
If $y \notin E$, then y is a limit point of E is what we want to show.

Since $y = \sup(E)$, $\exists\ x \in E \ni y - h < x < y$ for $h > 0$, otherwise $y-h$ would be an upper bound of E. But then y must be the limit point of E. $\Rightarrow y \in E' \subset \bar{E}$.
Hence in both the cases, we have $y \in \bar{E}$. $\blacksquare$

* Let $E = (a,b)$, $Y = \mathbb{R}^1$, $X = \mathbb{R}^2$.
Then E is open in $\mathbb{R}'$ but not in $\mathbb{R}^2$.
In general, let $E \subset Y \subset X$, where X is a metric space.

E open in X implies
if $p \in E$, $\exists\, r > 0 \ni d(p,q) < r,\ q \in X$
$$\Downarrow$$
$q \in E$.

E is open <u>relative to Y</u> if
for $p \in E$, $\exists\, r > 0 \ni d(p,q) < r,\ q \in Y$
$$\Downarrow$$
$q \in E$.

<u>Thm. 2.17</u> Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.

<u>Proof:</u> "$\Rightarrow$" Suppose E is open relative to Y. Let $p \in E$.
$\exists\, r_p > 0 \ni d(p,q) < r_p,\ q \in Y \Rightarrow q \in E$. —— (1)

Let $V_p = \{ q \in X : d(p,q) < r_p \}$. —— (2)
Let $G = \bigcup_{p \in E} V_p$. Obviously, G is open in X.
Note that $p \in V_p \ \forall\, p \in E$. Thus $E \subset \bigcup_{p \in E} V_p = G$

Since we are given $E \subset Y$, we have
$E \subset G \cap Y$ —— (3)

Note that from (1) and (2), $V_p \cap Y \subset E \ \forall\, p \in E$
$\Rightarrow \bigcup_{p \in E} (V_p \cap Y) \subset E$
But $\bigcup_{p \in E} (V_p \cap Y) = \left(\bigcup_{p \in E} V_p \right) \cap Y = G \cap Y$.

$\Rightarrow G \cap Y \subset E \longrightarrow (4)$

From (3) and (4), we conclude $E = G \cap Y$.

"$\Leftarrow$" Let G is open in X and $E = G \cap Y$.
Then by the defn. of an open set, for each $p \in E$, $\exists$ nbhd $V_p \ni V_p \subset G$

$\Rightarrow V_p \cap Y \subset G \cap Y$

$\Rightarrow V_p \cap Y \subset E$

$\Rightarrow$ for each $p \in E$, $\exists\, r_p > 0 \ni d(p,q) < r_p$, $q \in Y \Rightarrow q \in E$,

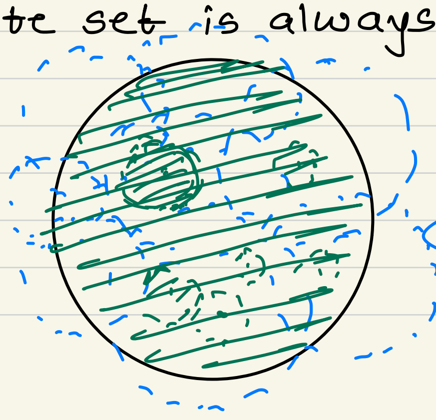
$\Rightarrow E$ is open relative to Y. $\blacksquare$

COMPACT SETS

Defn. Open cover : An open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of $X \ni E \subset \bigcup_\alpha G_\alpha$.

Defn. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

Eg. A finite set is always compact.



Suppose $K \subset \bigcup_\alpha G_\alpha$

Then K will be compact if $\exists$ $\alpha_1, \alpha_2, \ldots, \alpha_n \ni K \subset \bigcup_{i=1}^{n} G_{\alpha_i}$