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MA 509 – REAL ANALYSIS – LECTURE 16

Thm. 2.21 If $\{K_\alpha\}$ is a collection of compact subsets of a metric space such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\cap K_\alpha$ is non-empty.

Proof: Fix a member K_1 of the collection $\{K_\alpha\}$ and let $G_\alpha = K_\alpha^c$.

Suppose $\cap K_\alpha = \phi$, that means there isn't a single point lying in K_α for all α.

Without loss of generality, assume that no point of K_1 lies in $\{K_\alpha\}$. Then that means

$$K_1 \subset \bigcup_\alpha G_\alpha \quad , \text{ that is,}$$

$\{G_\alpha\}$ form an open cover of K_1. Since K_1 is compact, $\exists$ finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$ $\ni$ $K_1 \subset \bigcup_{i=1}^{n} G_{\alpha_i}$.

Then $K_1 \cap \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)^c = \phi$,

i.e. $K_1 \cap K_{\alpha_1} \cap K_{\alpha_2} \cap \dots \cap K_{\alpha_n} = \phi$, which is a contradiction to our hypothesis.

$$\Longrightarrow \bigcap_\alpha K_\alpha \neq \phi.$$

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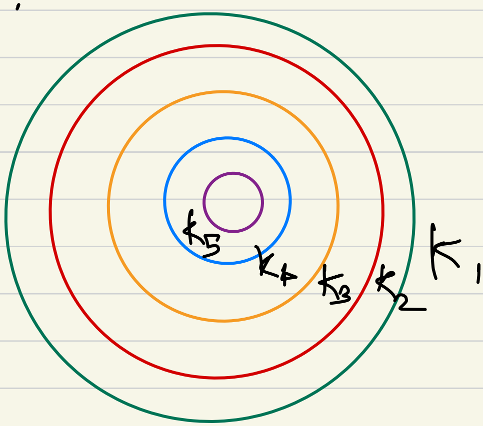
Cor. 2.22 If $\{K_n\}$ is a sequence of non-empty compact sets such that $K_n \supset K_{n+1}$, $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} K_n \neq \phi$.



Thm. 2.23 If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof: Suppose no point of K were a limit point of E, then <u>each</u> $q \in K$ has a nbhd V_q associated with it s.t. $V_q \cap E = \phi$ or $\{q\}$.

Then no finite subcollection of $\{V_q\}$ covers E, and so also K (since $E \subset K$).

But K is compact, hence contradiction.
$\Rightarrow$ E has a limit point in K.