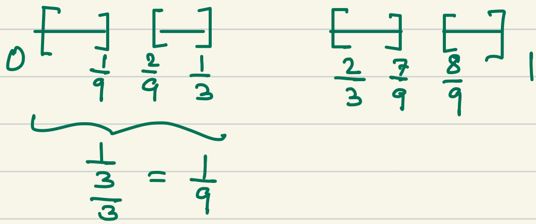
29/9/2020

MA 509 - REAL ANALYSIS - LECT. 20

CANTOR SET — An example of a perfect set in $\mathbb{R}$ with no open interval in it.

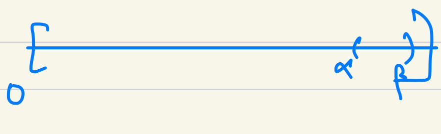
- Let $E_0 = [0,1]$.



- Remove $(\frac{1}{3}, \frac{2}{3})$ & let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

- Remove middle-thirds of each of these intervals & let

$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.



Continue this process indefinitely to get a seq. of compact sets E_n (since closed & bounded) $\ni$

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

Then note that by Cor. 2.22, we see that $P = \bigcap_{n=1}^{\infty} E_n \neq \phi$.

This P is called the Cantor set.

Properties of Cantor set

① Clearly, $\bigcap_{n=1}^{\infty} E_n$ is closed & bounded, hence compact.

② No segment of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$, for $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, has a non-empty intersection with P.

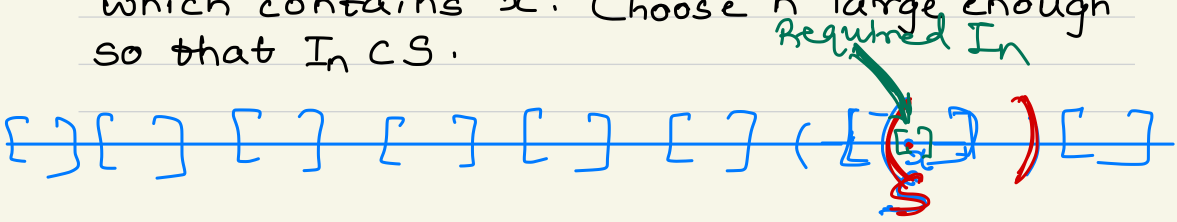
Now every open interval (α, β) has an open interval of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ inside it, say, if we choose $3^{-m} < \frac{\beta - \alpha}{6}$, we conclude that P contains no open interval (however small size it has)

③ P is perfect.

Proof: Suffices to show that P does not contain any isolated point.

Let $x \in P$ and let S be any open interval of $\mathbb{R}$ containing x.

Let I_n be that closed interval of E_n which contains x. Choose n large enough so that $I_n \subset S$.



Let x_n be an endpoint of $I_n \ni x_n \neq x$.
Then by construction of the Cantor set P, we have $x_n \in P$.

But S was any nbhd of x. Thus every nbhd of x intersects P in a point other than x.

$\Rightarrow$ x is a limit point of P.
$\Rightarrow$ P is Perfect.

(4) P is uncountable (by Thm. 2.29)

(5) P has measure zero.
Explanation: (Heuristic)
The total length removed from $[0,1]$ while constructing the Cantor set is given by

$$\sum_{n=0}^{\infty} 2^n \, 3^{-n-1}$$

total number of intervals in E_n

length of each such interval of E_n

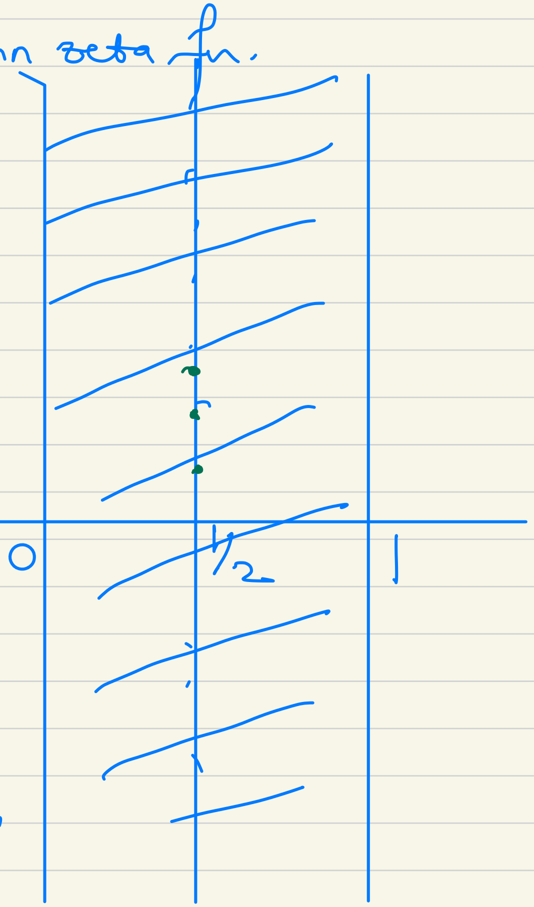
$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \, \frac{1}{1-2/3} = 1.$$

Hence the Cantor set has measure zero!

(6) P is nowhere dense (its closure has empty interior).

Riemann Hypothesis

$\zeta(s)$ Riemann zeta fn.



$0 < Re(s) < 1$

$Re(s) = \boxed{1/2}$

$\dots \ -8 \quad -6 \quad -4 \quad -2$

trivial zeros (roots)

G. H. Hardy (1914)

infinitely many complex zeros of $\zeta(s)$ have $Re(s) = 1/2$

Atle Selberg

Riemann (1859)

Newman

more than $\frac{1}{3}$ of complex zeros have $Re(s) = \frac{1}{2}$.

40.26%

41. ...