MA 509 – REAL ANALYSIS — LECTURE 22

6/10/20

Thm. 3.1 Let $\{p_n\}$ be a sequence in a metric space X.

(a) $\{p_n\} \to p \in X$ iff every nbhd of p contains all but finitely many of the terms of $\{p_n\}$

(b) If $p \in X$, $p' \in X$, and if $\{p_n\} \to p$ & $\{p_n\} \to p'$, then $p' = p$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in $E \ni \lim_{n\to\infty} p_n = p$.

Proof: (a) "$\Rightarrow$" Suppose $\{p_n\} \to p$ and V is any nbhd of p of radius ε

Claim: V contains all but finitely many terms of $\{p_n\}$.

To that end, note that by the defn. of a nbhd, if $q \in X$ is such that $d(p,q) < \varepsilon$, then $q \in V$.

Since $\{p_n\} \to p$, corresponding to this $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni \forall n \geq N$ $d(p_n, p) < \varepsilon$

$\Rightarrow p_n \in V \;\forall n \geq N$.

This proves the claim.

"$\Leftarrow$" Suppose every nbhd V of p contains all but finitely many terms of $\{p_n\}$.

Fix $\varepsilon > 0$. Let V be the nbhd of p containing all points $q \ni d(p,q) < \varepsilon$.

By assumption, $\exists N \in \mathbb{N}$ (corresponding to the above V) s.t. $p_n \in V$ if $n \geq N$,

$\Rightarrow d(p_n, p) < \varepsilon$ if $n \geq N$

$\Rightarrow \{p_n\} \to p$.

(b) If $p, p' \in X$ & $\{p_n\} \to p$ & $\{p_n\} \to p'$, then $p = p'$.

Proof:- Let $\varepsilon > 0$ be given. $\exists N, N' \in \mathbb{N} \ni$

$n \geq N \Rightarrow d(p_n, p) < \varepsilon/2$

$n \geq N' \Rightarrow d(p_n, p') < \varepsilon/2$

$\Rightarrow$ if $N^* = \max(N, N')$, then $\forall n \geq N^*$,

$$d(p,p') \leq d(p_n, p) + d(p_n, p') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we have $p = p'$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

Proof:

Spse $p_n \to p$. $\exists N \in \mathbb{N} \ni \forall n \geq N$, $d(p_n, p) < 1$.

Let $r = \max\{1, d(p_1,p), \ldots, d(p_N, p)\}$.
Then $d(p_n, p) \leq r \ \forall \ n \in \mathbb{N}$.
Hence $\{p_n\}$ is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in $E \ni p = \lim_{n\to\infty} p_n$.

Proof:- Every nbhd of p contains a point of E different from p.

Consider, in particular, the nbhds $B(p; 1/n), n \in \mathbb{N}$
& let $p_n \in B(p, 1/n)$
Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni N\varepsilon > 1$, i.e., $\frac{1}{N} < \varepsilon$.

If $n > N$, $\frac{1}{n} < \frac{1}{N} < \varepsilon$, so for such n, $d(p_n, p) < \varepsilon$.

$\Rightarrow p_n \to p$.

Thm. 3.2 Suppose $\{s_n\}, \{t_n\}$ are complex sequences & $\lim_{n\to\infty} s_n = s$ & $\lim_{n\to\infty} t_n = t$. Then,

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} c s_n = cs$, $\lim_{n\to\infty} (c + s_n) = c + s$, for any c.

(c) $\lim_{n\to\infty} s_n t_n = st$.

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$, $n \in \mathbb{N}$, and $s \neq 0$.

Proof:

(a), (b) exercise.

(c) Use $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t)$
$+ t(s_n - s)$. ——— (*)

Let $\varepsilon > 0$ be given. $\exists\, N_1, N_2 \in \mathbb{N} \ni$
$\forall n \geq N_1$, $|s_n - s| < \sqrt{\varepsilon}$ &
$\forall n \geq N_2$, $|t_n - t| < \sqrt{\varepsilon}$.
$\Rightarrow \forall n \geq \max(N_1, N_2)$, $|(s_n - s)(t_n - t)| < \varepsilon$.
Since $\varepsilon > 0$ is arbitrary, this implies that

$$\lim_{n \to \infty} (s_n - s)(t_n - t) = 0. \quad \text{——— (**)}$$

Hence from (*), (**) and the hypotheses, we have
$$\lim_{n \to \infty} (s_n t_n - st) = 0$$

Since $\lim_{n \to \infty} st = st$, from (a), we get

$$\lim_{n \to \infty} s_n t_n = st.$$

(d) Since $s_n \to s$, $\exists\, m \in \mathbb{N} \ni \forall n \geq m$,
$$|s_n - s| < \frac{1}{2}|s|.$$

Now reverse-Δ ineq., $\big||s_n| - |s|\big| < |s_n - s| < \frac{|s|}{2}$

$$\Rightarrow \quad -\frac{|s|}{2} < |s_n| - |s| < \frac{|s|}{2}$$

$$\Rightarrow \quad |s_n| > \frac{|s|}{2} \quad (n \geq m) \quad \text{——— (i)}$$

Now given $\varepsilon > 0$, $\exists N > m \ni \forall n \geq N$,

$$|s_n - s| < \frac{1}{2}|s|^2 \varepsilon. \quad \text{——— (ii)}$$

Hence, for $n \geq N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n||s|} < \frac{1}{2}|s|^2 \varepsilon \cdot \frac{2}{|s|} \cdot \frac{1}{|s|} = \varepsilon.$$

Thm. 3.3 (a) Suppose $\bar{x}_n \in \mathbb{R}^k$, $n \in \mathbb{N}$ &
$\bar{x}_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$.
Then $\{\bar{x}_n\} \to \{\bar{x}\}$ iff $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k)$

(b) Suppose $\{\bar{x}_n\}$, $\{\bar{y}_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a seq. of real numbers, &
$\bar{x}_n \to \bar{x}$, $\bar{y}_n \to \bar{y}$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty}(\bar{x}_n + \bar{y}_n) = \bar{x} + \bar{y}, \quad \lim_{n\to\infty} \bar{x}_n \cdot \bar{y}_n = \bar{x}\cdot\bar{y}$$

& $\lim_{n\to\infty} \beta_n \bar{x}_n = \beta x$.

Proof: Exercise.

SUBSEQUENCES

Defn. Let $\{p_n\}$ be a sequence.
Suppose $\{n_k\}$ be a sequence of positive integers s.t. $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$.

$n_1 = 500$, $n_2 = 671$, $n_3 = 786$, - - - - -
$\{p_{500}, p_{671}, p_{786}, \text{- - -}\}$

If $\{p_{n_i}\}$ converges, we call its limit a <u>subsequential limit</u> of $\{p_n\}$.

<u>Remark:</u> $\{p_n\} \to p$ iff every subsequence of $\{p_n\}$ converges to p.

$$\left\{ 1, 2+1, 1-\frac{1}{2}, 2+\frac{1}{2}, 1-\frac{1}{3}, 2+\frac{1}{3}, 1-\frac{1}{4}, 2+\frac{1}{4}, \ldots \right\}$$

<u>Thm 3.4</u> (i) Let $\{p_n\}$ be a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point in X.

(ii) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

$$\{ i^n, n \in \mathbb{N} \}$$

$$\{ i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, i \ldots \}$$