7/10/20

MA 509 – REAL ANALYSIS — LECTURE 23

Thm 3.4 (i) Let $\{p_n\}$ be a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point in X.

(ii) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof: (i) Let E be the range of $\{p_n\}$.

Case a) E is finite

In this case, $\exists\, p \in E$ & subsequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \cdots \ni p_{n_1} = p_{n_2} = \cdots = p$.

$\Rightarrow \{p_{n_i}\} \longrightarrow p$.

Case b) E is infinite

Then X compact implies E has a limit pt. in X, say p.

So $\exists\, n_1 \in \mathbb{N} \ni d(p, p_{n_1}) < 1$.

Having chosen $n_1, n_2, \dots, n_{i-1}$, we see that $\exists\, n_i \in \mathbb{N}$ $\ni n_i > n_{i-1}$ & $d(p, p_{n_i}) < \frac{1}{i}$.

(Note this follows because every nbhd of p intersects in infinitely many points of E.)

$\Longrightarrow \{p_{n_i}\} \to p$.

(ii) Note that every bdd. seq. of $\mathbb{R}^k$ lies in a compact subset of $\mathbb{R}^k$.

$\Rightarrow$ the result follows from (i).

<u>Thm. 3.5</u> Let X be a metric space. Then the set of all subsequential limits of a sequence $\{p_n\}$ in X is closed in X.

<u>Proof:</u> Let E^* be the set of all subsequential limits of $\{p_n\}$. Let q be a limit point of E^*.

<u>Claim:</u> $q \in E^*$.

Choose $n_1 \ni p_{n_1} \neq q$. Note that if such an n_1 didn't exist, then E^* is a singleton set, hence closed.

Let $\delta = d(p_{n_1}, q)$. Let $n_1, n_2, \dots, n_{i-1}$ be chosen. Since $q \in E^{*'}$, $\exists\, x \in E^* \ni$

$$d(x, q) < \frac{\delta}{2^i}.$$

But x is a limit point of E. So $\exists\, n_i \ni$ $n_i > n_{i-1}$ & $d(x, p_{n_i}) < \frac{\delta}{2^i}$.

Hence,
$$\begin{aligned} d(q, p_{n_i}) &\leq d(q, x) + d(x, p_{n_i}) \\ &< \frac{\delta}{2^i} + \frac{\delta}{2^i} \\ &= \frac{\delta}{2^{i-1}} \quad \text{for } i \in \mathbb{N}. \end{aligned}$$

$\Rightarrow \{p_{n_i}\} \to q$, so that $q \in E^*$.

CAUCHY SEQUENCES

Defn. A sequence $\{p_n\}$ in a metric space X is said to be Cauchy if for every $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$ $\forall n, m \geq N$, $d(p_n, p_m) < \varepsilon$.

- Is $\{p_n\} = \{\frac{1}{n}\}$ where $n \in \mathbb{N}$ Cauchy $\underline{\text{in } \mathbb{R}}$ / in $\mathbb{Q}$?

- Let $\mathbb{Q}$ be the metric space. For $n \in \mathbb{N}$, consider $\{p_n : p_n \in \mathbb{Q}$ and $p_n \to \sqrt{2}\}$. Is it Cauchy in $\mathbb{Q}$?

  Ans. Yes.
  But it is $\underline{\text{not}}$ convergent in $\mathbb{Q}$.

Defn. Let $E \subset X$ where X is a metric space. Let $S = \{d(p,q) : p \in E, q \in E\}$
Then $\text{diam}(E) = \sup(S)$, called the diameter of E.

Let $\{p_n\}$ be a sequence in X &
$E_N = \{p_m : m \in \mathbb{N}, m \geq N\}$.

Then $\{p_n\}$ is Cauchy iff $\lim_{N \to \infty} \text{diam}(E_N) = 0$.

$\{p_n\} \to p$ $\quad \varepsilon > 0$, $\exists N \in \mathbb{N} \ni \forall n \geq N$, $d(p_n, p) < \frac{\varepsilon}{2}$

$n \geq N$, $m \geq N$

$$d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thm. 3.6 Let X be a metric space. Then

(a) $\operatorname{diam}(\bar{E}) = \operatorname{diam}(E)$

(b) If K_n is a seq. of compact sets in X $\ni$ $K_n \supset K_{n+1}$, $n \in \mathbb{N}$, and if $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of a single point.

Proof: Since $E \subset \bar{E}$, obviously,

$$\operatorname{diam}(E) \leq \operatorname{diam}(\bar{E})$$

To show $\operatorname{diam}(\bar{E}) \leq \operatorname{diam}(E)$, fix $\varepsilon > 0$ & take $x, y \in \bar{E}$. Then $\exists$ $p, q \in E$ $\ni$ $d(p, x) < \frac{\varepsilon}{2}$ & $d(q, y) < \frac{\varepsilon}{2}$. Thus

$$\begin{aligned} d(x,y) &\leq d(x,p) + d(p,q) + d(q,y) \\ &< \frac{\varepsilon}{2} + d(p,q) + \frac{\varepsilon}{2} \\ &\leq \varepsilon + \operatorname{diam}(E) \end{aligned}$$

$\Rightarrow \operatorname{diam}(\bar{E}) \leq \varepsilon + \operatorname{diam}(E)$

Since ε was arbitrary,

$$\operatorname{diam}(\bar{E}) \leq \operatorname{diam}(E).$$

$$\Rightarrow \operatorname{diam}(\bar{E}) = \operatorname{diam}(E). \quad \boxtimes$$