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MA 509 – REAL ANALYSIS – LECT. 24

Thm. 3.6 Let X be a metric space. Then

(a) $\operatorname{diam}(\bar{E}) = \operatorname{diam}(E)$

(b) If K_n is a seq. of compact sets in X ∋ $K_n \supset K_{n+1}$, $n \in \mathbb{N}$, and if $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of a single point.

Proof: (a) Done last time.

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$. Then K is non-empty, by Cor. 2.22.

If K consists of more than one point, $\operatorname{diam}(K) > 0$.

But $K_n \supset K$ $\forall n \in \mathbb{N}$, so $\operatorname{diam}(K_n) \geq \operatorname{diam}(K)$ $\forall n \in \mathbb{N}$.

This contradicts the hypothesis that $\operatorname{diam}(K_n) \to 0$. ∎

Thm. 3.7 (a) In a metric space X, every convergent seq. is Cauchy.

(b) If X is a compact metric space, and if $\{p_n\}$ is a Cauchy sequence in X then $\{p_n\}$ converges to some point of X.

(c) In $\mathbb{R}^k$, every Cauchy sequence converges

Proof: (a) If $p_n \to p$, given an $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ $\ni$ $\forall\, n \geq N$, $d(p_n, p) < \varepsilon/2$.

Then $\forall\, m, n \geq N$,

$$d(p_n, p_m) \leq d(p_n, p) + d(p_m, p)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon.$$

$\Rightarrow$ $\{p_n\}$ is Cauchy.

(b) Let $\{p_n\}$ be a Cauchy sequence in the compact metric space X. For $N \in \mathbb{N}$, let

$$E_N = \{p_N, p_{N+1}, p_{N+2}, \ldots\}.$$

Then by defn. of diameter of a set, and part (a) of the previous theorem, we have

$$\lim_{N \to \infty} \operatorname{diam}(\overline{E_N}) = 0. \quad \text{——— (i)}$$

Each $\overline{E_N}$ is closed subset of the compact space X, hence compact. ——— (ii)

Also $E_N \supset E_{N+1}$ so that $\overline{E_N} \supset \overline{E_{N+1}}$ ——— (iii)

From (i), (ii) & (iii), $\bigcap_{n=1}^{\infty} \overline{E_n}$ is a singleton set; let

$p \in \overline{E_N}$ $\forall\, N \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. From (i), $\exists\, N_0 \in \mathbb{N}$ $\ni$ $\operatorname{diam}(\overline{E_N}) < \varepsilon$ for $N \geq N_0$.

Since $p \in \overline{E_N}$, $d(p, q) < \varepsilon$ for every $q \in \overline{E_N}$, and hence for every $q \in E_N$.

Thus, $d(p, p_n) < \varepsilon$ for $n \geq N_0$.
$\Rightarrow \{p_n\} \to p$. $\blacksquare$

(c) Let $\{\bar{p}_n\}$ be a Cauchy sequence in $\mathbb{R}^k$.
Suppose $E_N = \{\bar{p}_N, \bar{p}_{N+1}, \bar{p}_{N+2}, \ldots$ for $N \in \mathbb{N}$.

Since $\{\bar{p}_n\}$ is Cauchy, $\operatorname{diam} E_N \to 0$ as $N \to \infty$.
Hence $\exists\, N \ni \operatorname{diam}(E_N) < 1$.

The range of $\{\bar{p}_n\}$ is $E_N \cup \{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{N-1}\}$.
$\Rightarrow \{\bar{p}_n\}$ is bounded.

$d(\bar{p}_1, \bar{p}_N)$

$d(\bar{p}_3, \bar{p}_N) \leq d(\bar{p}_1, \bar{p}_3) + d(\bar{p}_1, \bar{p}_N)$

So can take, say,
$2\max\{1, d(\bar{p}_1, \bar{p}_2), \ldots, d(\bar{p}_{N-1}, \bar{p}_{N-2}), d(\bar{p}_1, \bar{p}_N)\}$ as the upper bound.

Since every bounded set of $\mathbb{R}^k$ has compact closure in $\mathbb{R}^k$, (c) now follows from (b).

COMPLETE METRIC SPACE

A metric space is said to be <u>complete</u> if every Cauchy sequence in it converges.

e.g. All compact metric spaces, $\mathbb{R}^k$ are complete.

<u>Remark</u> Every closed subset of a complete metric space is complete.
Proof: Let $\{p_n\}$ be a Cauchy sequence in the closed subset E of X. Thus it's a Cauchy seq.

in X, and hence converges to a point p in X. This p, being a limit point of the closed set E, then belongs to E, hence E is complete. $\boxtimes$

- <u>example</u> of a metric space which is not <u>complete</u>: $(\mathbb{Q}, d)$ with $d(x,y) = |x-y|$.