MA 509 - REAL ANALYSIS — LECTURE 25

19/10/2020

MONOTONIC SEQUENCES

Defn. A seq. $\{s_n\}$ of real numbers is said to be

(a) monotonically increasing if $s_n \le s_{n+1}$ $(n \in \mathbb{N})$

(b) monotonically decreasing if $s_n \ge s_{n+1}$ $(n \in \mathbb{N})$.

Thm 3.7 If $\{s_n\}$ is monotonic, then $\{s_n\}$ converges iff it is bounded.

Proof: We prove the above assertion for monot. incr. seq. The one for decreasing seq. is analogous.

Suppose $s_n \le s_{n+1}$ $\forall\, n \in \mathbb{N}$.

Let E be the range of $\{s_n\}$. If $\{s_n\}$ is bounded, let s be the least upper bound of E. Then

$$s_n \le s \quad (n \in \mathbb{N})$$

For every $\varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni$

$$s - \varepsilon < s_N \le s,$$

otherwise $s - \varepsilon$ would be an upper bound of E.

Since $\{s_n\}$ is increasing, for $n \ge N$,

$$s - \varepsilon < s_n \le s$$

So $\{s_n\} \to s$.

Convergence $\Longrightarrow$ bounded is already done. $\square$

UPPER AND LOWER LIMITS

Let $\{s_n\}$ be a sequence in $\mathbb{R}$ s.t. for every real M, $\exists\, N \in \mathbb{N} \ni n \ge N$ implies $s_n \ge M$.

Then we say $s_n \to +\infty$.

Similarly, if for every real M, $\exists\, N \in \mathbb{N} \ni n \geq N$ implies $s_n \leq M$, we say $s_n \to -\infty$.

LIMIT SUPERIOR AND LIMIT INFERIOR

Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

Let $E = \{ x : x \in \mathbb{R} \cup \{\pm\infty\},\ \exists$ a subsequence $\{s_{n_k}\}$ of $\{s_n\} \ni s_{n_k} \to x \}$.

Thus E consists of all subsequential limits of $\{s_n\}$ plus, possibly, $\pm\infty$.

Put $s^* = \sup(E)$ & $s_* = \inf(E)$

These numbers s^* and s_* are respectively called the limit superior and limit inferior of $\{s_n\}$ and are denoted by

$$\limsup_{n\to\infty} s_n = s^* \quad \text{and} \quad \liminf_{n\to\infty} s_n = s_* \qquad \underline{\underline{\text{OR}}}$$

by $\overline{\lim\limits_{n\to\infty}}\, s_n = s^*$ and $\underline{\lim\limits_{n\to\infty}}\, s_n = s_*$,

<u>Thm. 3.8</u> Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* be as defined above. Then s^* has following two properties:

(i) $s^* \in E$

(ii) If $x > s^*$, $\exists\, N \in \mathbb{N} \ni n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with (i) & (ii).

Analogous result holds for s_*.

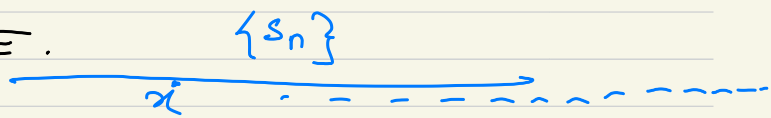
Proof: (i) If $s^* = +\infty$, E is not bounded above, hence so is $\{s_n\}$. This means there must be a subsequence $\{s_{n_k}\}$ of $\{s_n\} \ni s_{n_k} \to +\infty$.

$\Rightarrow s^* \in E$.

If $s^* \in \mathbb{R}$, then E is bounded above and also closed (from Thm. 3.5), so that $s^* \in E$ by a result proved earlier.

If $s^* = -\infty$, then $\sup(E) = -\infty$. Then E contains only one element $-\infty$, and there is no subsequential limit. Hence for any real M, $s_n > M$ for at most a finite number of values of n, so $s_n \to -\infty$.

$\Rightarrow s^* \in E$.



(ii) Suppose $\exists\, x > s^* \ni s_n \geq x$ for infinitely many values of n.

Then $\exists\, y \in E \ni y \geq x > s^*$. But $s^* = \sup(E)$

$\rightarrow\leftarrow$. Hence proved.

To show uniqueness of s^*, suppose there are 2 numbers s^* and s & suppose $s^* < s$. Then choose $x \ni s^* < x < s$. Since s^* satisfies (ii), $s_n < x$ for $n \geq N$ & N is some natural number. But then no subsequence can tend to s, which contradicts (i).

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Examples

a) $\{s_n\}$ is a sequence containing all rationals. Since rationals are dense in $\mathbb{R}$, i.e., $\overline{\{s_n\}} = \mathbb{R}$, so any $x \in \mathbb{R}$ is a subsequential limit. So

$$\limsup_{n\to\infty} s_n = +\infty, \quad \liminf_{n\to\infty} s_n = -\infty.$$

b) Let $s_n = \dfrac{(-1)^n}{1+1/n}$. Then

$$\limsup_{n\to\infty} s_n = +1 \quad \& \quad \liminf_{n\to\infty} s_n = -1$$

Suppose we take the subsequence $\{s_{2n}\}$ for $n \in \mathbb{N}$.

$$\{s_{2n}\} = \left\{\frac{1}{1+1/2n}\right\} \longrightarrow 1$$

Similarly,

$$\{s_{2n+1}\} = \left\{\frac{-1}{1+\frac{1}{2n+1}}\right\} \longrightarrow -1.$$

$$\limsup_{n\to\infty} s_n = \sup\{-1, 1\} = 1$$

$$\liminf_{n\to\infty} s_n = \inf\{-1, 1\} = -1.$$