MA 509 - REAL ANALYSIS — LECTURE 27   19/10/2020

Thm. 3.14 (COMPARISON TEST) (a) If $|a_n| \leq c_n$ for $n \geq N_0$, where $N_0 \in \mathbb{N}$ is fixed, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

(Note that (b) applies only to series of non-negative terms a_n.)

Proof: (a) Since $\sum c_n$ converges, given $\varepsilon > 0$, $\exists\, N \geq N_0$ $\ni$ $m \geq n \geq N$ implies $\sum_{k=n}^{m} c_k \leq \varepsilon$ by Cauchy criterion.

Therefore,
$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \varepsilon,$$
so that by Cauchy criterion $\sum a_n$ converges. $\boxtimes$

(b) Follows from (a) since if $\sum a_n$ converges, then (a) implies $\sum d_n$ should converge too. $\boxtimes$

e.g. (1) $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ & $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Hence (b) implies $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges.

(2) $\frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$ & $\sum \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$.

Series of non-negative terms

* Geometric series

Thm. 3.15 If $0 \le x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

If $x \geq 1$, the series diverges.

Proof: Let $x \neq 1$. Then

$$S_n := \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

Now using Thm. 3.10 (e), we see that for $|x| < 1$

$$\lim_{n\to\infty} S_n = \frac{1-0}{1-x} = \frac{1}{1-x}, \text{ otherwise } \lim_{n\to\infty} S_n = \infty.$$

Hence $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ for $|x| < 1$,

& diverges for $|x| > 1$.

Now for $x = 1$, $\sum_{n=0}^{\infty} x^n = 1 + 1 + 1 + \cdots$, which clearly diverges.

(CAUCHY CONDENSATION TEST)

Thm. 3.16 Suppose $a_1 \geq a_2 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Proof: Since both these series consist of non-negative terms, by Thm. 3.13, it suffices to show boundedness of partial sums. To that end, let

$a_1 + (a_2 + a_3) + \cdots (\; + a_n) + a_{n+1} + a_{n+2} + \cdots + a_{2^k - 1} + a_{2^k} + a_2$

$S_n = a_1 + a_2 + \cdots + a_n,$

$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$

- Then for $n < 2^k$,

$$\begin{aligned} s_n &\le a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \\ &\quad + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + (2a_2) + (4a_4) + \cdots + 2^k a_{2^k} \\ &= t_k . \end{aligned}$$

$\Rightarrow s_n \le t_k .$ ———— (a)

- For $n > 2^k$,

$$\begin{aligned} S_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2} a_1 + a_2 + (2a_4) + (4a_8) + \cdots + 2^{k-1} a_{2^k} \\ &= \frac{1}{2}(a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots 2^k a_{2^k}) \\ &= \frac{1}{2} t_k \end{aligned}$$

$\Rightarrow 2S_n \geq t_k .$ ———— (b)

From (a) & (b), the sequences $\{s_n\}$ & $\{t_k\}$ are either both bounded or both unbounded. ∎

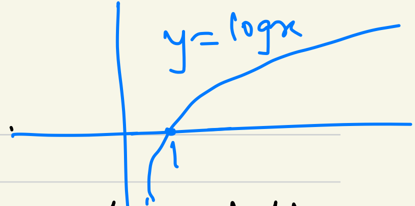
Thm. 3.17 $\sum \frac{1}{n^p}$ converges if $p > 1$ & diverges for $p \le 1$.

Proof :- If $p \le 0$, then $\frac{1}{n^p} \not\to 0$ as $n \to \infty$.

Hence $\sum \frac{1}{n^p}$ diverges.

If $p > 0$, then by Thm. 3.16, $\sum \frac{1}{n^p}$ converges iff $\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}}$ converges.

But $\displaystyle\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{k(1-p)}$.



Now $2^{1-p} < 1$ iff $1-p<0$, i.e; $p>1$, and then by Thm. 3.15, we see that $\displaystyle\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}}$ converges.

$\Rightarrow \sum \frac{1}{n^p}$ converges for $p>1$.

$\boxtimes$

<u>Thm. 3.18</u> If $p>1$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges; if $p \le 1$, it diverges.

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k (\log 2^k)^p}$$

<u>Proof</u>: $f(n) = \log n$ is an increasing fn. (proved later). Then $\left\{\frac{1}{n(\log n)^p}\right\}_{n=1}^{\infty}$ is a decreasing seq.

Hence by Thm. 3.16, the given series converges iff

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p} \text{ converges.}$$

However,

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k (\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p},$$

and so it converges if $p>1$, otherwise it diverges.

$\boxtimes$

<u>EULER NUMBER 'e'</u>

<u>Defn.</u> $\displaystyle e = \sum_{n=0}^{\infty} \frac{1}{n!}$, where $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \ge 1$, & $0! = 1$

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Note that $s_n = \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}$

$$= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2\ldots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$< 1 + \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right)$$

$$= 1 + \frac{1}{(1 - 1/2)} = 3.$$

Hence the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.