## • Least upper bound (supremum) and Greatest lower bound (infimum)

Suppose $S$ is an ordered set, $E \subset S$, and $E$ is bounded above, meaning $\exists\ \beta \in S \ni x \leq \beta$ for every $x \in E$. Suppose there exists an $\alpha \in S$ with the following properties:

(i) $\alpha$ is an upper bound of $E$.

(ii) If $\gamma < \alpha$ then $\gamma$ is not an upper bound of $E$.

Then $\alpha$ is called the least upper bound of $E$. (From (ii), it is easy to see that such an $\alpha$ is unique.)

Similarly if $E \subset S$ is bounded below, meaning $\exists\ \beta \in S \ni \beta \leq x\ \forall x \in E$. Suppose $\exists\ \alpha \in S$ with the following properties:

(i) $\alpha$ is a lower bound of $E$.

(ii) If $\beta > \alpha$, then $\beta$ is not a lower bound of $E$.

Then $\alpha$ is called the greatest lower bound of $E$. (Again, such an $\alpha$ is unique.)

Notation: • $\alpha = \sup(E)$ or $lub(E)$ in the first case

• $\alpha = \inf(E)$ or $glb(E)$ in the second case.

Examples: ① Consider again

$A = \{p \in \mathbb{Q} : p^2 < 2\}$ and $B = \{p \in \mathbb{Q} : p^2 > 2\}$.

Then $A$ is bounded above, and the upper bounds of $A$ are precisely the elements of $B$. Since $B$ contains no smallest member, $A$ has no least upper bound. in $\mathbb{Q}$.

Similarly, $B$ has no greatest lowerbound in $\mathbb{Q}$.

(2) Suppose $\alpha = \sup(E)$ exists. Still, $\alpha$ may or may not belong to $E$. For example, suppose

$$E_1 = \{r \in \mathbb{Q} : r < 0\} \quad \& \quad E_2 = \{r \in \mathbb{Q} : r \leq 0\}.$$

Then $\sup(E_1) = 0 = \sup(E_2)$. But $0 \notin E_1$, $0 \in E_2$.

Another example: Suppose $E = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\sup(E) = 1$ & $1 \in E$. But $\inf(E) = 0$, & $0 \notin E$.

## LEAST UPPER BOUND PROPERTY

An ordered set $S$ is said to have the least-upper-bound property if the following is true:

If $E \subset S$, $E \neq \phi$ and $E$ is bounded above, then $\sup(E)$ exists in $S$.

Thm. 1.1 Suppose $S$ is an ordered set with the least-upper-bound property, $B \subset S$, $B \neq \phi$ and $B$ is bounded below. Let $L$ be the set of all lower bounds of $B$. Then $\alpha = \sup(L)$ exists in $S$, and $\alpha = \inf(B)$.

In particular, $\inf(B)$ exists in $S$.

# MA 509 - Real Analysis (Lecture 2)

**Thm. 1.1** Suppose $S$ is an ordered set with the least-upper-bound property, $B \subset S$, $B \neq \phi$ and $B$ is bounded below. Let $L$ be the set of all lower bounds of $B$. Then $\alpha = \sup(L)$ exists in $S$, and $\alpha = \inf(B)$. In particular, $\inf(B)$ exists in $S$.

**Proof:** Since $B$ is bounded below (by hypothesis), we have $L \neq \phi$. Now $L$ consists of those $y \in S$ satisfying $y \leq x$ for every $x \in B$. This implies that

every $x \in B$ is an upper bound of $L$ —— (1)

Thus $L$ is bounded above, and non-empty. So by the lub-property, $L$ has a supremum in $S$, say $\alpha$, i.e.

$$\alpha = \sup(L).$$

If $\gamma < \alpha$, then $\gamma$ is not an upper bound of $L$ (for, otherwise, $\alpha$ would not be the least upper bound of $L$). Hence $\gamma \notin B$ (from 1), that is,

if $\gamma < \alpha$, then $\gamma \notin B$.

The contrapositive of this statement is

if $\gamma \in B$, then $\gamma \geqslant \alpha$, which, in turn, implies that $\alpha$ is a lower bound of $B$ so that

$$\alpha \in L.$$

Now if $\alpha < \beta$, then $\beta \notin L$ since $\alpha$ is an upper bound of $L$.

Thus $\alpha \in L$, but $\beta \notin L$ if $\beta > \alpha$. In other words, $\alpha$ is a lower bound of $B$, but $\beta$ is not if $\beta > \alpha$.

$$\Rightarrow \alpha = \inf(B).$$

$\boxtimes$

# FIELDS

A field is a set $F$ with two operations, addition & multiplication, which satisfy the following "field axioms".

(A) Axioms for addition

(A1) If $x, y \in F$, then $x+y \in F$.

(A2) (Commutativity): $x+y = y+x \ \forall x, y \in F$.

(A3) (Associativity): $(x+y)+z = x+(y+z) \ \forall x, y, z \in F$.

(A4) $\exists$ an element $0 \in F \ni 0+x = x \ \forall x \in F$.

(A5) Given $x \in F$, $\exists$ an element $y \in F \ni x+y=0$.
(This element $y$ is generally denoted by $-x$)

(M) Axioms for multiplication

(M1) If $x, y \in F$, then $xy \in F$

(M2) (Commutativity): $xy = yx \ \forall x, y \in F$

(M3) (Associativity): $(xy)z = x(yz) \ \forall x, y, z \in F$.

(M4) $\exists$ element $1 \in F$, $1 \neq 0 \ni 1x = x \ \forall x \in F$.

(M5) If $x \in F$ and $x \neq 0$, $\exists \ y \in F \ni xy = 1$
(This element $y$ is generally denoted by $1/x$)

(D) Distributivity

$\forall x, y, z \in F$, we have

$$x(y+z) = xy + xz.$$

Thm. 1.2 The axioms for additions imply

(a) If $x+y = x+z$, then $y=z$ (Cancellation Law)

(b) If $x+y = x$, then $y=0$ (Uniqueness of additive identity)

(c) If $x+y=0$, then $y=-x$ (Uniqueness of additive inverse)

(d) $-(-x) = x$